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Non-linear fluid dynamics of eccentric discs

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ABSTRACT
A new theory of eccentric accretion discs is presented. Starting from the basic fluid-dynamical equations in three dimensions, I derive the fundamental set of one-dimensional equations that describe how the mass, angular momentum and eccentricity vector of a thin disc evolve as a result of internal stresses and external forcing. The analysis is asymptotically exact in the limit of a thin disc, and allows for slowly varying eccentricities of arbitrary magnitude. The theory is worked out in detail for a Maxwellian viscoelastic model of the turbulent stress in an accretion disc. This generalizes the conventional alpha viscosity model to account for the non-zero relaxation time of the turbulence, and is physically motivated by a consideration of the nature of magnetohydrodynamic turbulence. It is confirmed that circular discs are typically viscously unstable to eccentric perturbations, as found by Lyubarskij, Postnov & Prokhorov, if the conventional alpha viscosity model is adopted. However, the instability can usually be suppressed by introducing a sufficient relaxation time and/or bulk viscosity. It is then shown that an initially uniformly eccentric disc does not retain its eccentricity as had been suggested by previous analyses. The evolutionary equations should be useful in many applications, including understanding the origin of planetary eccentricities and testing theories of quasi-periodic oscillations in X-ray binaries.

Key words: accretion, accretion discs – celestial mechanics – hydrodynamics – MHD – turbulence – waves.

1 INTRODUCTION

1.1 Background

In a thin accretion disc the stresses due to collective effects are relatively weak and the motion of the gas is nearly ballistic. Since the gravitational field is dominated by the central mass, the principal motion consists of Keplerian orbits. In the classical theory of accretion discs (e.g. Pringle 1981) these orbits are assumed to be circular and coplanar. Not only is this the simplest situation, it is also the expected outcome of the enhanced dissipation of energy that would result from initially misaligned or eccentric orbits.

Nevertheless, there are strong observational and theoretical grounds for believing that accretion discs in various situations are not flat but warped (e.g. Ogilvie 2000 and references therein). Such a situation can be caused by external forcing or can arise spontaneously through instabilities of an initially flat disc. Similarly, there are equally good reasons for investigating the possibility of eccentric discs. (The general case of a warped *and* eccentric disc deserves attention but is beyond the scope of the present paper.)

The recent discovery of extrasolar planets orbiting main-sequence stars (Mayor & Queloz 1995; Marcy, Cochran & Mayor 2000) is a major development in the history of astronomy. The distribution of orbital elements of the planets discovered to date poses some important challenges to theoretical models of planetary formation and dynamics. In particular, all of the planets with semi-major axes greater than 0.2 AU have significant eccentricities, several exceeding 0.5. One of the leading contenders for driving eccentricity is the tidal interaction between the planet and the disc from which it forms (e.g. Lubow & Artymowicz 2000).

A companion object on an eccentric orbit has a complicated tidal interaction with the disc. This is of considerable importance in connection with planetary rings and young binary systems, in addition to extrasolar planets. In the classical theory (Goldreich & Tremaine 1980), tightly wrapped density waves are launched at an array of resonances and propagate some way through the disc before dissipating. The resulting torques lead to evolution of the orbit of the companion and of the surface density of the disc. In addition, the companion may couple to global, low-frequency eccentric motions of the disc which are not described by the classical theory. The importance of such motions has been recognized by Tremaine (1998), and the methods required to understand them will be presented in this paper.

Further arguments support the notion of eccentric discs. Precessing eccentric discs are understood to exist in superhump binaries as the result of a tidal instability (e.g. Lubow 1991a and references therein). The general-relativistic apsidal precession of an eccentric inner disc is one of the leading explanations for kilohertz quasi-periodic oscillations in X-ray binaries (Stella, Vietri & Morsink 1999; Psaltis & Norman 2000; but see Marković & Lamb 2000 for a critical assessment). A transitory eccentric disc may be produced after tidal disruption of a star close to a black hole at the centre of a galaxy (Gurzadyan & Ozernoy 1979). Many-body systems with related dynamics include planetary rings, several of which have small but accurately measured eccentricities (e.g. Borderies, Goldreich & Tremaine 1983 and references therein), and the nucleus of the galaxy M31, for which Tremaine (1995) has proposed the model of an eccentric Keplerian disc.

The existing theoretical work on eccentric discs consists of analytical studies and numerical simulations, based almost exclusively on a two-dimensional description of the disc. Numerous authors have treated small eccentric perturbations of an initially circular disc as a special case of wave modes. Kato (1983) showed that global, low-frequency eccentric modes exist in an inviscid disc. A slow precession of the modes occurs owing to pressure forces. In a strictly inviscid disc, eccentric modes exist that have non-trivial vertical structure (Okazaki & Kato 1985), although these are not expected to be important when viscosity is included. A similar, modal description was used by Hirose & Osaki (1993) to describe superhumps in SU UMa binaries. Ostriker, Shu & Adams (1992) considered the near-resonant excitation of eccentric density waves in a self-gravitating disc due to a companion on an eccentric orbit. Lee & Goodman (1999) used novel techniques to analyse non-linear eccentric density waves in the tight-winding limit, for a self-gravitating disc.

Much effort has been devoted to explaining the superhump phenomenon in terms of a precessing, eccentric disc. Lubow (1991a) showed that a large-scale eccentric perturbation can grow through coupling with the tidal potential in a circular binary system. The coupling involves waves that are launched at eccentric Lindblad resonances, which are present only in tidally truncated discs in binaries of extreme mass ratio, as in SU UMa systems. Two-dimensional numerical simulations have been performed by Whitehurst (1988), Hirose & Osaki (1990), Whitehurst & King (1991), Lubow (1991b), Whitehurst (1994) and Murray (1996, 1998, 2000).

More general studies are of direct relevance to the present paper. Borderies et al. (1983) derived evolutionary equations for slightly eccentric planetary rings subject to quadrupolar and tidal forces, self-gravitation and viscosity. Syer & Clarke (1992, 1993) used the Gauss perturbation equations of celestial mechanics to argue that, contrary to the assumptions of classical accretion disc theory, an isolated viscous disc will retain its initial eccentricity indefinitely. Their numerical simulations supported this idea. Most recently, Lyubarskij, Postnov & Prokhorov (1994) derived evolutionary equations for a viscous accretion disc in which the orbits are ellipses sharing a common longitude of periastron. They confirmed the result of Syer & Clarke (1993) but only in the sense of a singular solution of the time-dependent equations, finding that an initially circular disc is viscously unstable to eccentric perturbations if the usual alpha viscosity model is adopted.

The aims of the present paper are to unify these previous treatments and to extend them in a number of important ways. A new set of evolutionary equations for an eccentric disc will be derived. The analysis of Lyubarskij et al. (1994) will be extended to allow for precession of the orbits, which can never be avoided and is the dominant feature in the pressure-driven modes of Kato (1983). The earlier modal description will be extended to allow for arbitrary eccentricities, and the effects of turbulent stresses and radiation. The analysis will also include important three-dimensional effects, which have been neglected in previous studies. In addition, the conventional alpha viscosity model will be generalized into a Maxwellian viscoelastic model, which is a physically motivated and more realistic description of the turbulent stress in an accretion disc.

1.2 Continuum celestial mechanics

As an introductory exercise, consider the problem of a test body orbiting in the gravitational field of a central object of mass M, but subject to a perturbing force per unit mass $\boldsymbol{f}$. Its equation of motion is

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}=-\frac{GM}{r^2}\,\boldsymbol{e}_r+\boldsymbol{f},\qquad(1)$$

where $\boldsymbol{u}=\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ is the velocity, with $\boldsymbol{r}=r\,\boldsymbol{e}_r$ being the position vector with respect to the central object. Since inclined orbits are beyond the scope of this paper, assume that the orbit and the perturbing force lie in the xy-plane.

The traditional method of determining the rate of change of the osculating orbital elements of the body involves evaluating the disturbing function and applying the Gauss perturbation equations of celestial mechanics (e.g. Brouwer & Clemence 1961). A more compact derivation uses the *eccentricity vector* $\boldsymbol{e}$, which lies in the plane of the orbit and may be defined by (Eggleton, Kiseleva & Hut 1998; Lynden-Bell 2000)

$$\boldsymbol{u}=\frac{GM}{h}\,\boldsymbol{e}_z\times(\boldsymbol{e}_r+\boldsymbol{e}),\qquad(2)$$

or, equivalently,

$$\boldsymbol{e}=-\frac{h}{GM}\,\boldsymbol{e}_z\times\boldsymbol{u}-\boldsymbol{e}_r,\qquad(3)$$

where

$$h\,\boldsymbol{e}_z=\boldsymbol{r}\times\boldsymbol{u}\qquad(4)$$

is the specific angular momentum. The position and velocity are then instantaneously equal to those of an elliptical Keplerian orbit of semi-latus rectum

$$\lambda=\frac{h^2}{GM}=r+\boldsymbol{r}\cdot\boldsymbol{e}\qquad(5)$$

and eccentricity $e=|\boldsymbol{e}|$, with $\boldsymbol{e}$ pointing towards periastron.

Substitution into the equation of motion (1) gives the relations

$$\frac{\mathrm{d}h}{\mathrm{d}t}\,\boldsymbol{e}_z=\boldsymbol{r}\times\boldsymbol{f},\qquad(6)$$

$$h\,\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}t}\,(\boldsymbol{e}_r+\boldsymbol{e})-\lambda\,\boldsymbol{e}_z\times\boldsymbol{f},\qquad(7)$$

which determine the evolution of the orbital elements and are equivalent to the Gauss perturbation equations.

The problem to be addressed in this paper is how to derive a related set of equations for a fluid disc in which $\boldsymbol{e} = \boldsymbol{e}(\lambda, t)$ varies not only with time but also from one orbit to the next. The evolutionary equations will be partial differential equations (PDEs) depending on one spatial variable, analogous to those obtained for warped discs (Pringle 1992; Ogilvie 1999, 2000), and equally suitable for semi-analytical or numerical implementation.

The perturbing force in a fluid disc includes a contribution from the internal stress in addition to any external forcing. The instantaneous elliptical Keplerian motion associated with the function $\boldsymbol{e}(\lambda, t)$ determines (part of) the velocity gradient tensor in the disc. An appropriate theory, satisfying the basic principles of continuum mechanics, is then required to relate the stress tensor to the velocity gradient tensor.

1.3 Modelling the turbulent stress

The simplest assumption, made in almost all theoretical work on accretion discs, is that the turbulent stress may be treated as an isotropic effective viscosity in the sense of the Navier-Stokes equation.[⋆] The stress tensor may then be written as

$$\mathbf{T} = \mu \left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\right] + (\mu_{\mathrm{b}} - \tfrac{2}{3}\mu)(\nabla\cdot\boldsymbol{u})\mathbf{1}, \qquad (8)$$

where μ is the effective (dynamic) shear viscosity and μ_{b} the effective bulk viscosity (which is often omitted). For the usual case of a steady, flat and circular Keplerian disc, this assumption suffices to generate the stress component $T_{R\phi}$ that is required for accretion to proceed. Together with the conventional parametrization of the viscosity coefficient, this constitutes the alpha viscosity model (Shakura & Sunyaev 1973), which was very successful in connecting theory and observations during a period when the theory was incomplete.

In recent years it has become widely accepted that the turbulent stress in most, if not all, accretion discs is magnetohydrodynamic (MHD) in origin, resulting from the non-linear development of the magnetorotational instability (Balbus & Hawley 1998). The turbulent stress tensor is

$$\mathbf{T} = \left\langle \frac{\boldsymbol{B}'\boldsymbol{B}'}{4\pi} - \rho\boldsymbol{u}'\boldsymbol{u}' - \rho'(\boldsymbol{u}\boldsymbol{u}' + \boldsymbol{u}'\boldsymbol{u} + \boldsymbol{u}'\boldsymbol{u}')\right\rangle, \qquad (9)$$

where the prime denotes a turbulent fluctuation from the mean value of a quantity, and the magnetic pressure fluctuation has been considered to be combined with the gas pressure fluctuation. Numerical simulations of the turbulence offer the possibility of measuring the stress and testing the validity of the alpha viscosity model in more general circumstances. In fact very few studies have been directed towards this aim (Abramowicz, Brandenburg & Lasota 1996; Torkelsson et al. 2000).

The most obvious deficiency of the alpha viscosity model is that it requires the stress to respond instantaneously to a change in the rate of strain. In an eccentric disc the strain field experienced by a fluid element changes during the course of each orbit. There is a strong suspicion that the assumption of an instantaneous response may be partially responsible for the eccentric instability described by Lyubarskij et al. (1994). In reality the turbulence is expected to have a non-zero relaxation time comparable (on dimensional grounds) to the orbital period.

It is therefore proposed that the stress tensor satisfies the equation

$$\mathbf{T} + \tau\mathcal{D}\mathbf{T} = \mu \left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\right] + (\mu_{\mathrm{b}} - \tfrac{2}{3}\mu)(\nabla\cdot\boldsymbol{u})\mathbf{1}, \qquad (10)$$

where τ is the relaxation time and $\mathcal{D}$ a linear operator defined by

$$\mathcal{D}\mathbf{T} = \partial_t\mathbf{T} + \boldsymbol{u}\cdot\nabla\mathbf{T} - (\nabla\boldsymbol{u})^{\mathrm{T}}\cdot\mathbf{T} - \mathbf{T}\cdot\nabla\boldsymbol{u} + 2(\nabla\cdot\boldsymbol{u})\mathbf{T}. \qquad (11)$$

This model (although with $\nabla\cdot\boldsymbol{u} = 0$ for an incompressible fluid) is used in rheology as a simple model of a viscoelastic medium, known as the *upper-convected Maxwell fluid*. The relaxation time was introduced by Clerk Maxwell (1866) in his study of the viscosity of gases. The Maxwellian viscoelastic fluid allows for a continuum of behaviour from an elastic solid (on time-scales short compared to τ) to a viscous fluid (on time-scales long compared to τ). The operator $\mathcal{D}$ is one of a family of convective derivative operators, acting on second-rank tensors, that satisfy material frame indifference, which is a fundamental principle of continuum mechanics. In the context of rheology, equation (10) can be justified either by abstract principles of continuum mechanics, or from a kinetic theory of polymer molecules that satisfy Hooke's law (Bird, Armstrong & Hassager 1987; Bird, Curtiss & Armstrong 1987).

There are good reasons to suppose that a viscoelastic model may offer a reasonable description of MHD turbulence in accretion discs. Magnetic field lines have tension and support Alfvén waves similar to waves on an elastic string. There is a close analogy between the tangled magnetic field lines in a turbulent medium and the tangled polymer molecules in a viscoelastic fluid. Indeed, the equation satisfied by the dyadic tensor $\boldsymbol{B}\boldsymbol{B}$ in ideal MHD is precisely[†]

$$\mathcal{D}(\boldsymbol{B}\boldsymbol{B}) = 0, \qquad (12)$$

and $\langle\boldsymbol{B}'\boldsymbol{B}'/4\pi\rangle$ is arguably the most important contribution to the total turbulent stress tensor (9).

This accounts for the 'elastic' aspect of MHD turbulence. The 'viscous' part of equation (10) can then be related informally to the (less well understood) dissipative and non-linear aspects of the turbulence. Viscoelastic models have been proposed for hydrodynamic turbulence (Crow 1968) but the case for MHD turbulence is arguably stronger.

This hypothesis is also related to the 'causal viscosity' model used in the context of accretion disc boundary layers to ensure that information is not propagated faster than the sound speed (Kley & Papaloizou 1997). In that context, only a single component of $\mathbf{T}$ is treated and the full expression for $\mathcal{D}$ is not used. Here, however, it is essential to treat the stress as a tensor and to ensure that the equations satisfy material frame indifference.

Note that one easily recovers the conventional alpha viscosity model by taking the limit $\tau \to 0$.

⋆ Note the crucial distinction between isotropy of the stress and isotropy of the effective viscosity.

† I am indebted to Prof. M. R. E. Proctor for pointing this out.

It is of interest to work out the implications of this model for a steady, flat and circular Keplerian disc. The horizontal stress components are predicted to be

$$T_{RR}=0,\qquad T_{R\phi}=-\frac{3}{2}\mu\Omega,\qquad T_{\phi\phi}=\frac{9}{2}\mathrm{We}\,\mu\Omega, \tag{13}$$

where Ω is the angular velocity and

$$\mathrm{We}=\Omega\tau \tag{14}$$

the *Weissenberg number*, which is expected to be of order unity. A comparison with the results of MHD simulations by Hawley, Gammie & Balbus (1995) suggests that the Maxwellian viscoelastic model performs rather well in predicting the turbulent stress tensor (9). The results of Table 2 in that paper imply that

$$\begin{aligned}&(T_{RR},T_{R\phi},T_{Rz},T_{\phi\phi},T_{\phi z},T_{zz})\\&\qquad\propto(0.105,-1,0.018,1.520,-0.050,0.076),\end{aligned} \tag{15}$$

whereas the viscoelastic model predicts

$$(T_{RR},T_{R\phi},T_{Rz},T_{\phi\phi},T_{\phi z},T_{zz})\propto(0,-1,0,3\mathrm{We},0,0). \tag{16}$$

Given that the numerical quantities are averaged over a limited time interval and involve some statistical error, the agreement is good and the results are consistent with $\mathrm{We}\approx 0.5$. It would be valuable to conduct further simulations designed to measure the relaxation time in a more direct way.

1.4 Plan of the paper

The remainder of this paper is organized as follows. In Section 2 I present a highly simplified analysis of an eccentric disc, with the purpose of explaining the principles of the method without the technical detail of the full model. In Section 3 I set out the basic equations and coordinate systems used in the rest of the paper. Section 4 contains the bulk of the analysis. I present an asymptotic development of the equations appropriate for a thin disc, reducing the problem to simpler units that are solved in turn to extract the required one-dimensional evolutionary equations. In Section 5 a linearized theory for small eccentricity is described and compared with the oversimple model of Section 2. An illustrative, time-dependent solution of the fully non-linear equations is given in Section 6. Finally, in Section 7, the results are summarized and a comparison with existing work is made. Recommendations for applications of this work are also given.

2 AN OVERSIMPLE MODEL

2.1 Analysis

In the following sections a detailed model of a thin, eccentric accretion disc will be developed, consisting of a fully non-linear asymptotic analysis of the fluid-dynamical equations in three dimensions, including dissipation of energy and radiative transport. Before setting out this theory, I present a grossly simplified model (two-dimensional, linear and barotropic) that is not recommended for further use but nevertheless illustrates some of the important principles without many of the technical details of the full calculation.

Suppose that the fluid is strictly two-dimensional and obeys the equation of mass conservation,

$$(\partial_t+\boldsymbol{u}\cdot\nabla)\rho=-\rho\nabla\cdot\boldsymbol{u}, \tag{17}$$

and the equation of motion,

$$\rho(\partial_t+\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\rho\nabla\Phi-\nabla p+\nabla\cdot\mathbf{T}, \tag{18}$$

where ρ is the density, $\boldsymbol{u}$ the velocity, p the pressure and $\mathbf{T}$ the turbulent stress tensor, which will be assumed to satisfy equation (10). The gravitational potential of the central mass M is $\Phi=-GM/R$, referred to plane polar coordinates (R,ϕ). The self-gravitation of the fluid is neglected.

Much use has been made of the analogy between the dynamics of a two-dimensional fluid and that of a thin disc. However, the analogy is not a formal one except under special circumstances (and never for an eccentric disc, as will be seen). In fact, a two-dimensional model corresponds to an infinite cylinder rather than a thin disc, although the gravitational potential cannot be interpreted in this sense. However, ρ, p and μ in the two-dimensional theory might be regarded as approximately equivalent to the corresponding vertically integrated quantities in the three-dimensional theory. To emphasize this, in this section the density will be written as Σ (surface density) instead of ρ. To avoid a consideration of the energy equation at this stage, suppose that the fluid is barotropic, so that $p=p(\Sigma)$, $\mu=\mu(\Sigma)$ and $\mu_{\mathrm{b}}=\mu_{\mathrm{b}}(\Sigma)$ are given functions.

Let (u,v) be the polar components of $\boldsymbol{u}$. Then one has

$$\left(\partial_t+u\partial_R+\frac{v}{R}\partial_\phi\right)\Sigma=-\frac{\Sigma}{R}\left[\partial_R(Ru)+\partial_\phi v\right], \tag{19}$$

$$\begin{aligned}\Sigma\left[\left(\partial_t+u\partial_R+\frac{v}{R}\partial_\phi\right)u-\frac{v^2}{R}\right]&=-\Sigma\frac{GM}{R^2}-\partial_R p\\&+\frac{1}{R}\partial_R(RT_{RR})+\frac{1}{R}\partial_\phi T_{R\phi}-\frac{T_{\phi\phi}}{R},\end{aligned} \tag{20}$$

$$\begin{aligned}\Sigma\left[\left(\partial_t+u\partial_R+\frac{v}{R}\partial_\phi\right)v+\frac{uv}{R}\right]&=-\frac{1}{R}\partial_\phi p\\&+\frac{1}{R^2}\partial_R(R^2T_{R\phi})+\frac{1}{R}\partial_\phi T_{\phi\phi},\end{aligned} \tag{21}$$

with

$$\begin{aligned}&T_{RR}+\tau\left\{\left(\partial_t+u\partial_R+\frac{v}{R}\partial_\phi\right)T_{RR}+\frac{2}{R}(u+\partial_\phi v)T_{RR}\right.\\&\qquad\left.-\frac{2}{R}(\partial_\phi u)T_{R\phi}\right\}=2\mu\partial_R u\\&\qquad+(\mu_{\mathrm{b}}-\tfrac{2}{3}\mu)\frac{1}{R}\left[\partial_R(Ru)+\partial_\phi v\right],\end{aligned} \tag{22}$$

$$\begin{aligned}&T_{R\phi}+\tau\left\{\left(\partial_t+u\partial_R+\frac{v}{R}\partial_\phi\right)T_{R\phi}-R\partial_R\left(\frac{v}{R}\right)T_{RR}\right.\\&\qquad\left.+\frac{1}{R}\left[\partial_R(Ru)+\partial_\phi v\right]T_{R\phi}-\frac{1}{R}(\partial_\phi u)T_{\phi\phi}\right\}\\&\qquad=\mu\left[R\partial_R\left(\frac{v}{R}\right)+\frac{1}{R}\partial_\phi u\right],\end{aligned} \tag{23}$$

$$\begin{aligned}&T_{\phi\phi}+\tau\left\{\left(\partial_t+u\partial_R+\frac{v}{R}\partial_\phi\right)T_{\phi\phi}-2R\partial_R\left(\frac{v}{R}\right)T_{R\phi}\right.\\&\qquad\left.+2(\partial_R u)T_{\phi\phi}\right\}=2\mu\left(\frac{u}{R}+\frac{1}{R}\partial_\phi v\right)\\&\qquad+(\mu_{\mathrm{b}}-\tfrac{2}{3}\mu)\frac{1}{R}\left[\partial_R(Ru)+\partial_\phi v\right].\end{aligned} \tag{24}$$

Let the basic state be an axisymmetric, circular disc that evolves on the slow, viscous time-scale. Adopt a system of units such that the radius of the disc and the characteristic orbital time-scale are $O(1)$. The slow evolution is captured by a slow time coordinate $T = \epsilon^2 t$, where the small parameter ϵ is a characteristic value of the angular semi-thickness H/R of the disc. For the basic state, introduce the expansions

$$u = \epsilon^2 u_2(R,T) + O(\epsilon^4), \tag{25}$$
$$v = R\Omega(R) + \epsilon^2 v_2(R,T) + O(\epsilon^4), \tag{26}$$
$$\Sigma = \Sigma_0(R,T) + O(\epsilon^2), \tag{27}$$
$$p = \epsilon^2 \left[p_0(R,T) + O(\epsilon^2)\right], \tag{28}$$
$$\mu = \epsilon^2 \left[\mu_0(R,T) + O(\epsilon^2)\right], \tag{29}$$
$$\mu_\mathrm{b} = \epsilon^2 \left[\mu_{\mathrm{b}0}(R,T) + O(\epsilon^2)\right], \tag{30}$$
$$\mathbf{T} = \epsilon^2 \left[\mathbf{T}_0(R,T) + O(\epsilon^2)\right], \tag{31}$$

where

$$\Omega = \left(\frac{GM}{R^3}\right)^{1/2} \tag{32}$$

is the Keplerian angular velocity. Substitution into equations (19)–(21) leads to

$$(\partial_T + u_2\partial_R)\Sigma_0 = -\frac{\Sigma_0}{R}\partial_R(Ru_2), \tag{33}$$
$$-2\Sigma_0\Omega v_2 = -\partial_R p_0 - \frac{T_{\phi\phi0}}{R}, \tag{34}$$
$$\frac{1}{2}\Sigma_0\Omega u_2 = \frac{1}{R^2}\partial_R(R^2 T_{R\phi0}), \tag{35}$$

with

$$T_{RR0} = 0, \tag{36}$$
$$T_{R\phi0} = -\frac{3}{2}\mu_0\Omega, \tag{37}$$
$$T_{\phi\phi0} = \frac{9}{2}\mathrm{We}\,\mu_0\Omega, \tag{38}$$

and this leads to the well known evolutionary equation for the surface density (e.g. Pringle 1981),

$$\partial_T\Sigma_0 = \frac{3}{R}\partial_R\left[R^{1/2}\partial_R\left(R^{1/2}\mu_0\right)\right]. \tag{39}$$

Now consider slowly varying, one-armed linear perturbations of the basic state, such that the Eulerian perturbation of u, say, is

$$\mathrm{Re}\left[u'(R,T)\,\mathrm{e}^{-\mathrm{i}\phi}\right]. \tag{40}$$

For the perturbations, introduce the expansions

$$u' = u'_0(R,T) + \epsilon^2 u'_2(R,T) + O(\epsilon^4), \tag{41}$$
$$v' = v'_0(R,T) + \epsilon^2 v'_2(R,T) + O(\epsilon^4), \tag{42}$$
$$\Sigma' = \Sigma'_0(R,T) + O(\epsilon^2), \tag{43}$$
$$p' = \epsilon^2 \left[p'_0(R,T) + O(\epsilon^2)\right], \tag{44}$$
$$\mu' = \epsilon^2 \left[\mu'_0(R,T) + O(\epsilon^2)\right], \tag{45}$$
$$\mu'_\mathrm{b} = \epsilon^2 \left[\mu'_{\mathrm{b}0}(R,T) + O(\epsilon^2)\right], \tag{46}$$
$$\mathbf{T}' = \epsilon^2 \left[\mathbf{T}'_0(R,T) + O(\epsilon^2)\right]. \tag{47}$$

In this linear analysis, the overall amplitude of the perturbations is of course arbitrary, and the above scaling is chosen for convenience.

Equations (20) and (21) at $O(1)$ yield

$$\Sigma_0(-\mathrm{i}\Omega u'_0 - 2\Omega v'_0) = 0, \tag{48}$$
$$\Sigma_0(-\mathrm{i}\Omega v'_0 + \frac{1}{2}\Omega u'_0) = 0, \tag{49}$$

with the general solution

$$u'_0 = \mathrm{i}R\Omega E, \tag{50}$$
$$v'_0 = \frac{1}{2}R\Omega E, \tag{51}$$

where $E(R,T)$ is a complex function to be determined. It is easily verified that this solution corresponds to a small eccentric perturbation, with complex eccentricity $E = e\,\mathrm{e}^{\mathrm{i}\omega}$ corresponding to eccentricity $e(R,T)$ and longitude of periastron $\omega(R,T)$. Equivalently, $E = e_x + \mathrm{i}e_y$, where (e_x, e_y) are the Cartesian components of the eccentricity vector.

Equation (19) at $O(1)$ then yields the density perturbation,

$$\Sigma'_0 = R\partial_R(\Sigma_0 E). \tag{52}$$

The pressure and viscosity perturbations follow from the barotropic relations, e.g.

$$p'_0 = \left(\frac{\mathrm{d}p}{\mathrm{d}\Sigma}\right)\Sigma'_0. \tag{53}$$

Equations (20) and (21) at $O(\epsilon^2)$ contain u'_2 and v'_2, but these may be eliminated by taking an appropriate linear combination. This results in an evolutionary equation for E, which may be written in the form

$$\begin{aligned}2\Sigma_0 R\Omega\partial_T E = {} & -2\Sigma_0 u_2\partial_R(R\Omega E) - \Omega\partial_R(R\Sigma_0 u_2 E)\\ & -2\mathrm{i}R^{1/2}\Omega\partial_R(R^{1/2}\Sigma_0 v_2 E) + \frac{\mathrm{i}}{R^2}\partial_R(R^2 p'_0)\\ & -\frac{\mathrm{i}}{R}\partial_R(RT'_{RR0}) + 2R^{-3/2}\partial_R(R^{3/2}T'_{R\phi0}) - \frac{\mathrm{i}T'_{\phi\phi0}}{R}.\end{aligned} \tag{54}$$

The stress perturbations satisfy the equations

$$\begin{aligned}&T'_{RR0} + \mathrm{We}(-\mathrm{i}T'_{RR0} - 2T_{R\phi0}E)\\ &\quad = 2\mathrm{i}\mu_0\partial_R(R\Omega E) + \mathrm{i}(\mu_{\mathrm{b}0} - \tfrac{2}{3}\mu_0)R\Omega\partial_R E,\end{aligned} \tag{55}$$

$$\begin{aligned}&T'_{R\phi0} + \mathrm{We}\left[\frac{3}{2}T'_{RR0} - \mathrm{i}T'_{R\phi0} + \mathrm{i}R\partial_R(T_{R\phi0}E) - T_{\phi\phi0}E\right]\\ &\quad = -\frac{3}{2}\left(\frac{\partial\mu}{\partial\Sigma}\right)R\Omega\partial_R(\Sigma_0 E) + \frac{1}{2R}\mu_0\partial_R(R^2\Omega E),\end{aligned} \tag{56}$$

$$\begin{aligned}&T'_{\phi\phi0} + \mathrm{We}\left[3T'_{R\phi0} - \mathrm{i}T'_{\phi\phi0} - R^{5/2}\partial_R(R^{-3/2}E)T_{R\phi0}\right.\\ &\quad \left.+\mathrm{i}RE\partial_R T_{\phi\phi0} + 2\mathrm{i}R^{3/2}\partial_R(R^{-1/2}E)T_{\phi\phi0}\right]\\ &\quad = \mathrm{i}\mu_0\Omega E + \mathrm{i}(\mu_{\mathrm{b}0} - \tfrac{2}{3}\mu_0)R\Omega\partial_R E.\end{aligned} \tag{57}$$

Thus one has obtained evolutionary equations for the surface density (equation 39) and the complex eccentricity variable (equation 54). In this linear theory the equations are decoupled because the small eccentricity makes a negligible change to the stresses that cause accretion.

2.2 Evolution of eccentricity in a purely viscous fluid

Equation (54) may be written in the form of a linear evolutionary equation for the eccentricity,

$$2\Sigma_0 R^2\partial_T E = \mathcal{A}R^2\partial_R^2 E + \mathcal{B}R\partial_R E + \mathcal{C}E. \tag{58}$$

The general form of the complex coefficients $(\mathcal{A},\mathcal{B},\mathcal{C})$ is complicated, but in the limit of a purely viscous disc, $\mathrm{We}\to 0$, one obtains

$$\mathcal{A}=3\mu-3\Sigma\left(\frac{\mathrm{d}\mu}{\mathrm{d}\Sigma}\right)+(\mu_\mathrm{b}-\tfrac{2}{3}\mu)+\frac{\mathrm{i}\Sigma}{\Omega}\left(\frac{\mathrm{d}p}{\mathrm{d}\Sigma}\right), \tag{59}$$

$$\mathcal{B}=\frac{12}{R\Omega}\partial_R(R^2\Omega\mu)-3R\partial_R\mu-3\partial_R\left(R\Sigma\frac{\mathrm{d}\mu}{\mathrm{d}\Sigma}\right)+\frac{1}{R^2\Omega}\partial_R\left[(\mu_\mathrm{b}-\tfrac{2}{3}\mu)R^3\Omega\right]+\frac{\mathrm{i}}{R^2\Omega}\partial_R\left(R^3\Sigma\frac{\mathrm{d}p}{\mathrm{d}\Sigma}\right), \tag{60}$$

$$\mathcal{C}=-2R\partial_R\mu+\frac{\mathrm{i}R}{\Omega}\partial_R p, \tag{61}$$

where the subscript zeros have been omitted.

First, suppose that an initial condition is specified in which $E=\text{constant}$, so that the eccentricity is uniform and the ellipses are aligned. The initial rate of change of E is determined by the coefficient $\mathcal{C}$. In a steady disc far from the inner radius, μ is nearly independent of R and the real part of $\mathcal{C}$ nearly vanishes. Therefore there is no initial viscous evolution of eccentricity, as has been argued by Syer & Clarke (1992, 1993) and Lyubarskij et al. (1994). However, the imaginary part of $\mathcal{C}$ does not vanish (except in the unlikely case that the pressure is independent of R) and causes a differential precession of the orbits. Viscosity will then act on the twisted configuration. It follows that a steady, uniformly eccentric disc is *not* a solution of the model. This difference comes about because the calculations of Syer & Clarke and Lyubarskij et al. neglected the differential precession caused by slightly non-Keplerian rotation resulting from the radial pressure gradient.

Now consider a limit in which the eccentricity varies on a length-scale $\ell\ll R$. (One requires $\ell\gg H$, however, in order that the ordering scheme adopted above remain valid.) Then equation (54) becomes approximately

$$\partial_T E\approx\frac{1}{2\Sigma}\left[3\mu-3\Sigma\left(\frac{\mathrm{d}\mu}{\mathrm{d}\Sigma}\right)+(\mu_\mathrm{b}-\tfrac{2}{3}\mu)+\frac{\mathrm{i}\Sigma}{\Omega}\left(\frac{\mathrm{d}p}{\mathrm{d}\Sigma}\right)\right]\partial_R^2E. \tag{62}$$

This is a linear diffusion equation with a complex diffusion coefficient. The imaginary part of the coefficient, which is due to pressure, results in wave-like propagation of the eccentricity and to the modes studied by Kato (1983). The real part determines the viscous diffusion of eccentricity. If the real part of the diffusion coefficient is negative, instability occurs. *Within this oversimple model*, therefore, an initially circular disc is unstable to developing eccentricity on small scales if

$$\frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln\Sigma}>1+\frac{1}{3}\left(\frac{\mu_\mathrm{b}}{\mu}-\frac{2}{3}\right), \tag{63}$$

i.e. if the vertically integrated viscosity increases sufficiently rapidly with increasing surface density. This criterion agrees with the findings of Lyubarskij et al. (1994) if the bulk term (the term in brackets) is neglected. However, this analysis shows that a sufficiently large bulk viscosity stabilizes the disc.

Interestingly, equation (63) is identical to the criterion for viscous overstability of short-wavelength *axisymmetric* modes in a two-dimensional disc (Kato 1978; see Willerding 1998 for the effect of bulk viscosity). This should not be surprising because the local properties of waves in a thin disc are almost independent of the azimuthal wavenumber when the radial wavelength is much shorter than the azimuthal wavelength. A related result was also found by Goldreich & Tremaine (1978), who examined the damping effect of shear and bulk viscosity on density waves in planetary rings. They considered the case in which μ and μ_b were constant, but commented that viscosity might produce growth rather than decay if μ and μ_b were to depend on Σ. Note that the condition (63) is not related to the criterion for thermal-viscous instability, $\mathrm{d}\ln\mu/\mathrm{d}\ln\Sigma<0$.

In the alpha models (Shakura & Sunyaev 1973) for gas-pressure dominated discs, $\partial\ln\mu/\partial\ln\Sigma$ is equal to $5/3$ for a Thomson opacity law and $10/7$ for a Kramers opacity law. Stability therefore requires μ_b/μ to exceed $8/3$ or $41/21$, respectively. Although bulk viscosity is not often discussed in the context of alpha models, there is no reason to suppose that accretion disc turbulence has a vanishing effective bulk viscosity. In addition, it is known that radiation damping, which has been neglected here, can mimic a bulk viscosity (Papaloizou & Pringle 1977). If the instability does occur, however, it will produce short-wavelength variations that invalidate the theory developed here. The non-linear outcome would have to be followed using numerical simulations that allowed for the viscosity to be a function of the surface density. The non-linear development of the axisymmetric version of the instability has been computed by, e.g., Papaloizou & Stanley (1986) and Kley, Papaloizou & Lin (1993).

Starting from equation (54) it is possible to derive a conservation law for eccentricity in the form‡

$$\partial_T(\Sigma R^2\Omega|E|^2)=\frac{1}{2R}\partial_R\left[\Sigma\frac{\mathrm{d}p}{\mathrm{d}\Sigma}R^3(\mathrm{i}E^*\partial_RE-\mathrm{i}E\partial_RE^*)\right]+\text{viscous terms}. \tag{64}$$

This shows that, in the absence of viscosity, the eccentricity is conserved in a certain sense. The conserved quantity is related to the 'angular momentum deficit' which is conserved in the secular theory of celestial mechanics (e.g. Laskar 1997). The eccentricity flux due to pressure effects is similar to the particle flux in Schrödinger's equation. The viscous terms are not necessarily negative definite, indicating the possibility of instability as described above.

2.3 Effect of a non-zero relaxation time

The general form of the coefficient $\mathcal{A}$ for a Maxwellian viscoelastic fluid is

$$\mathcal{A}=(1-\mathrm{iWe})^{-1}\left\{3\mu-3\Sigma\left(\frac{\mathrm{d}\mu}{\mathrm{d}\Sigma}\right)+(\mu_\mathrm{b}-\tfrac{2}{3}\mu)-\left(\frac{3\mathrm{iWe}}{1-\mathrm{iWe}}\right)\left[(1+\mathrm{iWe})\mu+(\mu_\mathrm{b}-\tfrac{2}{3}\mu)\right]\right\}+\frac{\mathrm{i}\Sigma}{\Omega}\left(\frac{\mathrm{d}p}{\mathrm{d}\Sigma}\right). \tag{65}$$

Instability on small scales occurs if $\mathrm{Re}(\mathcal{A})<0$, and one finds

$$(1+\mathrm{We}^2)^2\mathrm{Re}(\mathcal{A})=3(1+4\mathrm{We}^2-\mathrm{We}^4)\mu-3(1+\mathrm{We}^2)\Sigma\left(\frac{\mathrm{d}\mu}{\mathrm{d}\Sigma}\right)+(1+7\mathrm{We}^2)(\mu_\mathrm{b}-\tfrac{2}{3}\mu). \tag{66}$$

‡ I am indebted to Dr S. H. Lubow for suggesting this calculation.

For either Thomson or Kramers opacity, the instability persists for all values of We if $\mu_{\rm b}/\mu = 0$. However, for reasonable values of the bulk viscosity and relaxation time, the instability can be quenched. For example, the case $\mu_{\rm b}/\mu = 2/3$ corresponds to the vanishing of the 'bulk term' in equation (10). For this value, the instability is quenched for $3^{-1/2} < \mathrm{We} < 2^{1/2}$ (i.e. $0.577 < \mathrm{We} < 1.414$) in the case of Thomson opacity, and for $0.423 < \mathrm{We} < 1.547$ in the case of Kramers opacity. It is tantalizing that the value tentatively deduced from simulations of MHD turbulence, $\mathrm{We} \approx 0.5$, may or may not be sufficient to stabilize the disc, and the outcome may differ according to the opacity law.

A related result is that the viscous overstability of axisymmetric waves can be quenched by introducing a more sophisticated turbulence model with a non-zero relaxation time (Kato 1994). However, a direct comparison with Kato's work is not possible. His second-order closure model is aimed at modelling the turbulent Reynolds stress in an incompressible fluid that undergoes a notional hydrodynamic instability, whereas the Maxwellian viscoelastic model used in the present paper is aimed at modelling (predominantly) the turbulent Maxwell stress in a compressible fluid that undergoes the magnetorotational instability.

3 THE FULL MODEL

3.1 Basic equations

In this section the basic equations governing a fluid disc in three dimensions are expressed in a general, time-independent coordinate system. Following the usual notation of tensor calculus, g_{ab} denotes the metric tensor and ∇_a the covariant derivative.

The equation of mass conservation is

$$(\partial_t + u^a\nabla_a)\rho = -\rho\nabla_a u^a, \tag{67}$$

where ρ is the density and $\boldsymbol{u}$ the velocity. The equation of motion is

$$\rho(\partial_t + u^b\nabla_b)u^a = -\rho\nabla^a\Phi - \nabla^a p + \nabla_b T^{ab} + \rho f^a, \tag{68}$$

where Φ is the gravitational potential, p the pressure, $\mathbf{T}$ the turbulent stress tensor and $\boldsymbol{f}$ the external force per unit mass.

Under the assumption that the turbulent stress behaves similarly to a Maxwellian viscoelastic fluid, the stress tensor satisfies equation (10), i.e.

$$\begin{aligned}T^{ab} + \tau\left[(\partial_t + u^c\nabla_c)T^{ab} - T^{ac}\nabla_c u^b - T^{bc}\nabla_c u^a + 2(\nabla_c u^c)T^{ab}\right]\\ = \mu(\nabla^a u^b + \nabla^b u^a) + (\mu_{\rm b} - \tfrac{2}{3}\mu)(\nabla_c u^c)g^{ab},\end{aligned} \tag{69}$$

where τ is the relaxation time, μ the effective shear viscosity and $\mu_{\rm b}$ the effective bulk viscosity. The energy equation is

$$\begin{aligned}\left(\frac{1}{\gamma-1}\right)(\partial_t + u^a\nabla_a)p = -\left(\frac{\gamma}{\gamma-1}\right)p\nabla_a u^a\\ +T^{ab}\nabla_a u_b - \nabla_a F^a_{\rm rad},\end{aligned} \tag{70}$$

where γ is the adiabatic exponent and $\boldsymbol{F}_{\rm rad}$ the radiative energy flux, given in the Rosseland approximation for an optically thick medium by

$$F^a_{\rm rad} = -\frac{16\sigma T^3}{3\kappa\rho}\nabla^a T, \tag{71}$$

where σ is the Stefan-Boltzmann constant, T the temperature and κ the opacity. The equation of state of an ideal gas,

$$p = \frac{k\rho T}{\mu_{\rm m} m_{\rm H}}, \tag{72}$$

is adopted, where k is Boltzmann's constant, $\mu_{\rm m}$ the mean molecular weight and $m_{\rm H}$ the mass of the hydrogen atom. The opacity is assumed to be of the power-law form

$$\kappa = C_\kappa \rho^x T^y, \tag{73}$$

where C_κ is a constant. This includes the important cases of Thomson scattering opacity ($x = y = 0$) and Kramers opacity ($x = 1$, $y = -7/2$).

The gravitational potential is $\Phi = -GM/r$, where r is the distance from the central object. The self-gravitation of the disc is neglected. The effective viscosity coefficients are assumed to be given by an alpha parametrization. The precise form of the alpha prescription relevant to an eccentric disc is to some extent debatable. It will be convenient to adopt the form

$$\mu = \alpha p\left(\frac{GM}{\lambda^3}\right)^{-1/2}, \qquad \mu_{\rm b} = \alpha_{\rm b} p\left(\frac{GM}{\lambda^3}\right)^{-1/2}, \tag{74}$$

where α and $\alpha_{\rm b}$ are the dimensionless shear and bulk viscosity parameters, and λ is the semi-latus rectum of the elliptical orbits (introduced below). In the limit of a circular disc, this prescription reduces to the usual one, $\mu = \alpha p/\Omega$, etc., where Ω is the orbital angular velocity. Finally, the Weissenberg number is defined by

$$\mathrm{We} = \tau\left(\frac{GM}{\lambda^3}\right)^{1/2}, \tag{75}$$

where τ is the relaxation time. Again, this reduces to $\mathrm{We} = \Omega\tau$ in the limit of a circular disc.

3.2 Orbital coordinates

Consider a thin, eccentric accretion disc in which the dominant motion consists of coplanar, elliptical Keplerian orbits. Let (R, ϕ, z) be cylindrical polar coordinates such that the mid-plane of the disc corresponds to $z = 0$. Then the shape of the orbits is given by

$$R = \lambda(1 + e\cos\theta)^{-1}, \tag{76}$$

where λ is the semi-latus rectum, e the eccentricity and $\theta = \phi - \omega$ the azimuth relative to periastron (known as the 'true anomaly' in celestial mechanics). The semi-latus rectum is related to the specific angular momentum $h = (GM\lambda)^{1/2}$ and is conveniently adopted as a label for the different orbits. In general the eccentricity and longitude of periastron may vary continuously from one orbit to the next and also with time, so that $e = e(\lambda, t)$ and $\omega = \omega(\lambda, t)$. This is illustrated in Fig. 1. Let $\dot{e}$ and e' then denote $\partial_t e$ and $\partial_\lambda e$, etc. It will also be convenient to define the complex eccentricity $E = e\,\mathrm{e}^{\mathrm{i}\omega}$.

The analysis of an eccentric disc is greatly facilitated by the introduction of *orbital coordinates* (λ, ϕ) in place of (R, ϕ). However, some preliminary analysis must be carried out because these coordinates are both non-orthogonal and time-dependent.

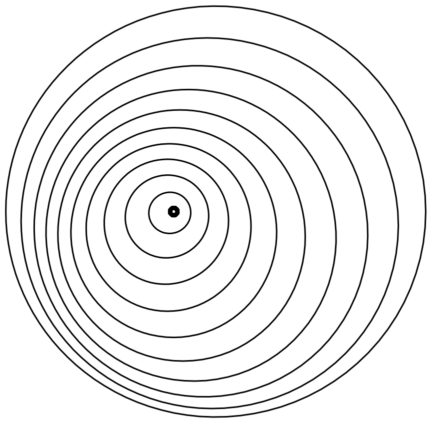

Figure 1. Example of instantaneous orbits in an eccentric disc.

3.2.1 Spatial derivatives

To evaluate the spatial derivatives in the governing equations, one must obtain the metric coefficients and connection components. The two-dimensional line element is given by

$$\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\phi^2 = (R_\lambda\,\mathrm{d}\lambda + R_\phi\,\mathrm{d}\phi)^2 + R^2\,\mathrm{d}\phi^2, \tag{77}$$

where the subscripts on R stand for partial derivatives of the function $R(\lambda,\phi,t)$. These can be evaluated from equation (76), but the more general notation will be retained. Thus the metric coefficients are

$$g_{\lambda\lambda} = R_\lambda^2, \qquad g_{\lambda\phi} = R_\lambda R_\phi, \qquad g_{\phi\phi} = R^2 + R_\phi^2. \tag{78}$$

As expected, the square root of the metric determinant is equal to the Jacobian of the coordinate system:

$$g^{1/2} = J = \frac{\partial(x,y)}{\partial(\lambda,\phi)} = \frac{\partial(x,y)}{\partial(R,\phi)}\frac{\partial(R,\phi)}{\partial(\lambda,\phi)} = RR_\lambda. \tag{79}$$

The inverse metric coefficients are

$$g^{\lambda\lambda} = \frac{R^2+R_\phi^2}{R^2R_\lambda^2}, \qquad g^{\lambda\phi} = -\frac{R_\phi}{R^2R_\lambda}, \qquad g^{\phi\phi} = \frac{1}{R^2}. \tag{80}$$

The components of the metric connection, given by

$$\Gamma^a_{bc} = \tfrac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}), \tag{81}$$

are found to be

$$\Gamma^\lambda_{\lambda\lambda} = \frac{R_{\lambda\lambda}}{R_\lambda}, \qquad \Gamma^\lambda_{\lambda\phi} = \frac{R_{\lambda\phi}}{R_\lambda} - \frac{R_\phi}{R}, \tag{82}$$

$$\Gamma^\lambda_{\phi\phi} = -\frac{(R^2+2R_\phi^2-RR_{\phi\phi})}{RR_\lambda}, \tag{83}$$

$$\Gamma^\phi_{\lambda\lambda} = 0, \qquad \Gamma^\phi_{\lambda\phi} = \frac{R_\lambda}{R}, \qquad \Gamma^\phi_{\phi\phi} = \frac{2R_\phi}{R}. \tag{84}$$

The coordinate system is trivially extended to three dimensions by incorporating the vertical coordinate z. With the exception of $g_{zz} = g^{zz} = 1$, all metric coefficients and connection components involving z vanish. The divergences of a vector and of a symmetric tensor in three dimensions are then

$$\begin{aligned}\nabla_a u^a &= \partial_a u^a + \Gamma^a_{ba}u^b, &(85)\\ \nabla_b T^{ab} &= \partial_b T^{ab} + \Gamma^a_{cb}T^{cb} + \Gamma^b_{cb}T^{ac}. &(86)\end{aligned}$$

The vector divergence can be written explicitly as

$$\frac{1}{J}\partial_\lambda(Ju^\lambda) + \frac{1}{J}\partial_\phi(Ju^\phi) + \partial_z u^z, \tag{87}$$

while the λ-, ϕ- and z-components of the tensor divergence are

$$\frac{1}{JR_\lambda}\partial_\lambda(JR_\lambda T^{\lambda\lambda}) + \frac{R^2}{JR_\lambda^2}\partial_\phi\left(\frac{JR_\lambda^2}{R^2}T^{\lambda\phi}\right) \\ -\frac{1}{RR_\lambda}(R^2+2R_\phi^2-RR_{\phi\phi})T^{\phi\phi} + \partial_z T^{\lambda z}, \tag{88}$$

$$\frac{1}{JR^2}\partial_\lambda(JR^2T^{\lambda\phi}) + \frac{1}{JR^2}\partial_\phi(JR^2T^{\phi\phi}) + \partial_z T^{\phi z}, \tag{89}$$

$$\frac{1}{J}\partial_\lambda(JT^{\lambda z}) + \frac{1}{J}\partial_\phi(JT^{\phi z}) + \partial_z T^{zz}, \tag{90}$$

respectively.

For a physically meaningful solution, the eccentricity is restricted in magnitude. In order that the orbits be closed, one obviously requires $|E| < 1$. A further restriction is that the Jacobian determinant should remain positive, which implies $|E - \lambda E'| < 1$. If this condition were violated, neighbouring orbits would intersect each other.

3.2.2 Time-derivatives

The equations of Section 3.1 are valid only in a time-independent ('inertial') coordinate system. To adapt them to orbital coordinates $q^a = (\lambda,\phi,z)$, observe that the partial derivatives at fixed inertial and orbital coordinates are related by

$$(\partial_t)_{\mathrm{inertial}} = (\partial_t)_{\mathrm{orbital}} + \dot{q}^a\partial_a, \tag{91}$$

where

$$\dot{q}^a = (\partial_t)_{\mathrm{inertial}}q^a. \tag{92}$$

Thus, for a scalar quantity such as ρ in equation (67) or p in equation (70), one replaces

$$\partial_t\rho \mapsto \partial_t\rho + \dot{q}^a\partial_a\rho. \tag{93}$$

When differentiating the vector $\boldsymbol{u}$ in equation (68) one must also take into account the time-dependence of the orbital basis vectors. One should then replace

$$\partial_t u^a \mapsto \partial_t u^a + \dot{q}^b\partial_b u^a - u^b\partial_b\dot{q}^a. \tag{94}$$

This is most easily derived by noting that $u^\lambda = \boldsymbol{u}\cdot\nabla\lambda$, etc., and obtaining the equation for the time-derivative of this quantity. More generally, one is replacing ∂_t with $\partial_t + \pounds_{\dot{\boldsymbol{q}}}$, where $\pounds$ denotes the Lie derivative. Thus in equation (69) one should replace

$$\partial_t T^{ab} \mapsto \partial_t T^{ab} + \dot{q}^c\partial_c T^{ab} - T^{cb}\partial_c\dot{q}^a - T^{ac}\partial_c\dot{q}^b. \tag{95}$$

Hereafter ∂_t will represent the derivative at fixed orbital coordinates. For orbital coordinates only $\dot\lambda$ is non-zero, so $\dot{q}^a\partial_a = \dot\lambda\partial_\lambda$. The identity

$$\partial_t J + \partial_\lambda(J\dot\lambda) \tag{96}$$

will be used below. This is established by noting that $(\partial_t)_{\rm inertial}R = 0$, so that

$$\partial_t R + \dot{\lambda} R_\lambda = 0, \tag{97}$$

and so

$$\partial_t J + \partial_\lambda(J\dot{\lambda}) = \partial_t(RR_\lambda) - \partial_\lambda(R\partial_t R) = 0, \tag{98}$$

as required.

4 ASYMPTOTIC DEVELOPMENT

4.1 Scaled variables and expansions

The key to analysing this problem, as with many dynamical problems in accretion discs, is to recognize that there is a separation of scales. The disc is thin, in the sense that the semi-thickness H of the disc satisfies $H/R \ll 1$. Additionally, the evolution of surface density and eccentricity occurs on a time-scale much longer than the orbital time-scale of the fluid. These ideas lead naturally to the introduction of scaled variables and an asymptotic development.

Let the small parameter ϵ be a characteristic value of the angular semi-thickness H/R of the disc. Adopt a system of units such that the radius of the disc and the characteristic orbital time-scale are $O(1)$. Then define the stretched vertical coordinate $\zeta = z/\epsilon$, which is $O(1)$ inside the disc. The slow evolution is captured by a slow time coordinate $T = \epsilon^2 t$ and one may replace

$$e(\lambda,t) \mapsto e(\lambda,T), \qquad \omega(\lambda,t) \mapsto \omega(\lambda,T). \tag{99}$$

For the fluid variables, introduce the expansions

$$\begin{aligned}
u^\lambda &= \epsilon^2 u_2^\lambda(\lambda,\phi,\zeta,T) + O(\epsilon^4), && (100)\\
u^\phi &= \Omega(\lambda,\phi,T) + \epsilon^2 u_2^\phi(\lambda,\phi,\zeta,T) + O(\epsilon^4), && (101)\\
u^z &= \epsilon u_1^z(\lambda,\phi,\zeta,T) + \epsilon^3 u_3^z(\lambda,\phi,\zeta,T) + O(\epsilon^5), && (102)\\
\rho &= \epsilon^s\left[\rho_0(\lambda,\phi,\zeta,T) + \epsilon^2\rho_2(\lambda,\phi,\zeta,T) + O(\epsilon^4)\right], && (103)\\
p &= \epsilon^{s+2}\left[p_0(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (104)\\
T^{\lambda\lambda} &= \epsilon^{s+2}\left[T_0^{\lambda\lambda}(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (105)\\
T^{\lambda\phi} &= \epsilon^{s+2}\left[T_0^{\lambda\phi}(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (106)\\
T^{\phi\phi} &= \epsilon^{s+2}\left[T_0^{\phi\phi}(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (107)\\
T^{\lambda z} &= \epsilon^{s+3}\left[T_1^{\lambda z}(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (108)\\
T^{\phi z} &= \epsilon^{s+3}\left[T_1^{\phi z}(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (109)\\
T^{zz} &= \epsilon^{s+2}\left[T_0^{zz}(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (110)\\
\mu &= \epsilon^{s+2}\left[\mu_0(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (111)\\
\mu_{\rm b} &= \epsilon^{s+2}\left[\mu_{\rm b0}(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (112)\\
T &= \epsilon^2\left[T_0(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (113)\\
F_{\rm rad}^z &= \epsilon^{s+3}\left[F_0(\lambda,\phi,\zeta,T) + O(\epsilon^2)\right], && (114)
\end{aligned}$$

while the horizontal components of $\boldsymbol{F}_{\rm rad}$ are $O(\epsilon^{s+4})$. Here s is a positive parameter, which drops out of the analysis, although formally one requires $s = (4-2y)/(2+x)$ in order to balance powers of ϵ in the opacity law. Note that the dominant motion is an orbital motion with angular velocity Ω, independent of ζ.

The potential is expanded in a Taylor series,

$$\Phi = \Phi_0(\lambda,\phi,T) + \tfrac{1}{2}\epsilon^2\zeta^2\Phi_2(\lambda,\phi,T) + O(\epsilon^4), \tag{115}$$

where

$$\Phi_0 = -\frac{GM}{R}, \qquad \Phi_2 = \frac{GM}{R^3}. \tag{116}$$

The external force per unit mass is also expanded as

$$\begin{aligned}
f^\lambda &= \epsilon^2\left[f_0^\lambda(\lambda,\phi,T) + O(\epsilon^2)\right], && (117)\\
f^\phi &= \epsilon^2\left[f_0^\phi(\lambda,\phi,T) + O(\epsilon^2)\right], && (118)\\
f^z &= O(\epsilon^3). && (119)
\end{aligned}$$

This is the correct form if $\boldsymbol{f}$ represents the tidal force due to a companion object in a coplanar orbit. The ϵ^2 scaling will be seen to be appropriate, and indicates that the external force is small compared to the gravitational force of the central object.

When these expansions are substituted into the equations of Section 2, various equations are obtained at different orders in ϵ. The required equations comprise three sets, representing three different physical problems, and these will be considered in turn.

4.2 Orbital motion

The horizontal components of the equation of motion (68) at leading order $[O(\epsilon^s)]$ are

$$\rho_0\Gamma^\lambda_{\phi\phi}\Omega^2 = -\rho_0(g^{\lambda\lambda}\partial_\lambda\Phi_0 + g^{\lambda\phi}\partial_\phi\Phi_0), \tag{120}$$

$$\rho_0(\Omega\partial_\phi\Omega + \Gamma^\phi_{\phi\phi}\Omega^2) = -\rho_0(g^{\lambda\phi}\partial_\lambda\Phi_0 + g^{\phi\phi}\partial_\phi\Phi_0). \tag{121}$$

Since Φ_0 is a function of R only, these simplify to

$$(R^2 + 2R_\phi^2 - RR_{\phi\phi})\Omega^2 = R\frac{\mathrm{d}\Phi_0}{\mathrm{d}R}, \tag{122}$$

$$\partial_\phi(R^2\Omega) = 0, \tag{123}$$

and are satisfied by the angular velocity of an elliptical Keplerian orbit,

$$\Omega = \left(\frac{GM}{\lambda^3}\right)^{1/2}(1 + e\cos\theta)^2, \tag{124}$$

where $e(\lambda,T)$ and $\omega(\lambda,T)$ are arbitrary. It is useful to note from the above that

$$R^2 + 2R_\phi^2 - RR_{\phi\phi} = \frac{R^3}{\lambda}, \tag{125}$$

which simplifies the expression for one of the connection components,

$$\Gamma^\lambda_{\phi\phi} = -\frac{R^2}{\lambda R_\lambda}. \tag{126}$$

The orbital period is[§]

$$P = \int\frac{\mathrm{d}\phi}{\Omega} = 2\pi(1-e^2)^{-3/2}\left(\frac{\lambda^3}{GM}\right)^{1/2}. \tag{127}$$

[§] Throughout, integrals with respect to ϕ are carried out from 0 to 2π, and integrals with respect to ζ over the full vertical extent of the disc.

4.3 Slow velocities and time-evolution

The equation of mass conservation (67) at leading order $[O(\epsilon^s)]$ is

$$(\Omega\partial_\phi+u_1^z\partial_\zeta)\rho_0=-\rho_0\left[\frac{1}{J}\partial_\phi(J\Omega)+\partial_\zeta u_1^z\right]. \tag{128}$$

Introduce the surface density at leading order $[O(\epsilon^{s+1})]$,

$$\tilde\Sigma(\lambda,\phi,T)=\int\rho_0\,\mathrm{d}\zeta. \tag{129}$$

Then the vertically integrated version of equation (128) may be written in the form

$$\partial_\phi(J\tilde\Sigma\Omega)=0, \tag{130}$$

which determines the variation of surface density around the orbit due to eccentricity. It will be convenient to introduce a pseudo-circular surface density,

$$\Sigma(\lambda,T)=\frac{1}{2\pi\lambda}\int J\tilde\Sigma\,\mathrm{d}\phi, \tag{131}$$

which has the property that the mass contained between orbits λ_1 and λ_2 is

$$2\pi\int_{\lambda_1}^{\lambda_2}\Sigma\,\lambda\,\mathrm{d}\lambda. \tag{132}$$

It will also be useful to introduce the second vertical moment of the density,

$$\tilde{\mathcal{I}}(\lambda,\phi,T)=\int\rho_0\zeta^2\,\mathrm{d}\zeta, \tag{133}$$

and the pseudo-circular average,

$$\mathcal{I}(\lambda,T)=\frac{1}{2\pi\lambda}\int J\tilde{\mathcal{I}}\,\mathrm{d}\phi. \tag{134}$$

The equation of mass conservation (67) at $O(\epsilon^{s+2})$ is

$$\begin{aligned}&\left[\partial_T+(u_2^\lambda+\dot\lambda)\partial_\lambda+u_2^\phi\partial_\phi+u_3^z\partial_\zeta\right]\rho_0+(\Omega\partial_\phi+u_1^z\partial_\zeta)\rho_2\\&\quad=-\rho_0\left[\frac{1}{J}\partial_\lambda(Ju_2^\lambda)+\frac{1}{J}\partial_\phi(Ju_2^\phi)+\partial_\zeta u_3^z\right]\\&\quad\quad-\rho_2\left[\frac{1}{J}\partial_\phi(J\Omega)+\partial_\zeta u_1^z\right].\end{aligned} \tag{135}$$

The horizontal components of the equation of motion (68) at $O(\epsilon^{s+2})$ are

$$\begin{aligned}&\rho_0\left[(\Omega\partial_\phi+u_1^z\partial_\zeta)u_2^\lambda-\Omega\partial_\phi\dot\lambda+2\Gamma^\lambda_{\lambda\phi}\Omega u_2^\lambda+2\Gamma^\lambda_{\phi\phi}\Omega u_2^\phi\right]\\&\quad=-\frac{\rho_0\zeta^2}{2R_\lambda}\frac{\mathrm{d}\Phi_2}{\mathrm{d}R}-g^{\lambda\lambda}\partial_\lambda p_0-g^{\lambda\phi}\partial_\phi p_0\\&\quad\quad+\frac{1}{JR_\lambda}\partial_\lambda(JR_\lambda T_0^{\lambda\lambda})+\frac{R^2}{JR_\lambda^2}\partial_\phi\left(\frac{JR_\lambda^2}{R^2}T_0^{\lambda\phi}\right)\\&\quad\quad-\frac{R^2}{\lambda R_\lambda}T_0^{\phi\phi}+\partial_\zeta T_1^{\lambda z}+\rho_0 f_0^\lambda,\end{aligned} \tag{136}$$

$$\begin{aligned}&\rho_0\left[\partial_T+(u_2^\lambda+\dot\lambda)\partial_\lambda+u_2^\phi\partial_\phi\right]\Omega\\&\quad+\rho_0\left[(\Omega\partial_\phi+u_1^z\partial_\zeta)u_2^\phi+2\Gamma^\phi_{\lambda\phi}\Omega u_2^\lambda+2\Gamma^\phi_{\phi\phi}\Omega u_2^\phi\right]\\&\quad=-g^{\lambda\phi}\partial_\lambda p_0-g^{\phi\phi}\partial_\phi p_0+\frac{1}{JR^2}\partial_\lambda(JR^2T_0^{\lambda\phi})\\&\quad\quad+\frac{1}{JR^2}\partial_\phi(JR^2T_0^{\phi\phi})+\partial_\zeta T_1^{\phi z}+\rho_0 f_0^\phi.\end{aligned} \tag{137}$$

It will be sufficient to work with the vertically integrated versions of equations (135), (136) and (137), which may be written in the form

$$\begin{aligned}&(\partial_T+\dot\lambda\partial_\lambda)\tilde\Sigma+\frac{1}{J}\partial_\lambda\int J\rho_0u_2^\lambda\,\mathrm{d}\zeta\\&\quad+\frac{1}{J}\partial_\phi\int J(\rho_0u_2^\phi+\rho_2\Omega)\,\mathrm{d}\zeta=0,\end{aligned} \tag{138}$$

$$\Omega\partial_\phi(v^\lambda-\dot\lambda)+2\Gamma^\lambda_{\lambda\phi}\Omega v^\lambda+2\Gamma^\lambda_{\phi\phi}\Omega v^\phi=A^\lambda, \tag{139}$$

$$\begin{aligned}&\left[\partial_T+(v^\lambda+\dot\lambda)\partial_\lambda+v^\phi\partial_\phi\right]\Omega+\Omega\partial_\phi v^\phi\\&\quad+2\Gamma^\phi_{\lambda\phi}\Omega v^\lambda+2\Gamma^\phi_{\phi\phi}\Omega v^\phi=A^\phi,\end{aligned} \tag{140}$$

where $\boldsymbol{v}(\lambda,\phi,T)$ is a mean horizontal velocity defined by

$$\tilde\Sigma v^a=\int\rho_0u_2^a\,\mathrm{d}\zeta, \tag{141}$$

and $\boldsymbol{A}(\lambda,\phi,T)$ a mean horizontal acceleration given by

$$\begin{aligned}\tilde\Sigma A^\lambda&=-\frac{\tilde{\mathcal{I}}}{2R_\lambda}\frac{\mathrm{d}\Phi_2}{\mathrm{d}R}+\frac{1}{JR_\lambda}\partial_\lambda(JR_\lambda\mathcal{T}^{\lambda\lambda})\\&\quad+\frac{R^2}{JR_\lambda^2}\partial_\phi\left(\frac{JR_\lambda^2}{R^2}\mathcal{T}^{\lambda\phi}\right)-\frac{R^2}{\lambda R_\lambda}\mathcal{T}^{\phi\phi}+\tilde\Sigma f_0^\lambda,\end{aligned} \tag{142}$$

$$\tilde\Sigma A^\phi=\frac{1}{JR^2}\partial_\lambda(JR^2\mathcal{T}^{\lambda\phi})+\frac{1}{JR^2}\partial_\phi(JR^2\mathcal{T}^{\phi\phi})+\tilde\Sigma f_0^\phi, \tag{143}$$

where

$$\mathcal{T}^{ab}=\int(T_0^{ab}-p_0g^{ab})\,\mathrm{d}\zeta \tag{144}$$

is the vertically integrated stress tensor including the pressure term.

The objective at this stage is to extract evolutionary equations for Σ and E without attempting to obtain a complete solution for v^λ and v^ϕ. First, equation (138) is integrated over ϕ to eliminate the mass flux around the orbit and obtain

$$\partial_T\int J\tilde\Sigma\,\mathrm{d}\phi+\partial_\lambda\int J\tilde\Sigma(v^\lambda+\dot\lambda)\,\mathrm{d}\phi=0, \tag{145}$$

where the identity (96) has been used. This expresses the conservation of mass in one dimension and involves the *relative velocity* $(v^\lambda+\dot\lambda)$. Let $v(\lambda,T)$ be a mean accretion velocity defined by

$$v\int J\tilde\Sigma\,\mathrm{d}\phi=\int J\tilde\Sigma(v^\lambda+\dot\lambda)\,\mathrm{d}\phi. \tag{146}$$

Then equation (145) may be written in the form

$$\dot\Sigma+\frac{1}{\lambda}\partial_\lambda(\lambda v\Sigma)=0, \tag{147}$$

formally identical to the equation of mass conservation for a circular disc.

Next, equations (139) and (140) may be rewritten in the form

$$\Omega\partial_\phi\left(\frac{R_\lambda^2}{R^2}v^\lambda\right)-\frac{2R_\lambda}{\lambda}\Omega v^\phi=\frac{R_\lambda^2}{R^2}B^\lambda, \tag{148}$$

$$\Omega\partial_\phi(R^2v^\phi)+v^\lambda\frac{\mathrm{d}h}{\mathrm{d}\lambda}=R^2B^\phi, \tag{149}$$

where $h=(GM\lambda)^{1/2}$ is the specific angular momentum and

$$B^\lambda=A^\lambda+\Omega\partial_\phi\dot\lambda,\qquad B^\phi=A^\phi-\frac{\dot\lambda}{R^2}\frac{\mathrm{d}h}{\mathrm{d}\lambda}. \tag{150}$$

Equation (149) expresses the conservation of angular momentum and may be integrated over the orbit to obtain

$$\frac{\mathrm{d}h}{\mathrm{d}\lambda}\int J\tilde{\Sigma}(v^\lambda+\dot{\lambda})\,\mathrm{d}\phi=\int J\tilde{\Sigma}R^2A^\phi\,\mathrm{d}\phi, \tag{151}$$

which involves exactly the same average relative velocity as in equation (145). Indeed, these two equations clearly relate the evolution of mass to the total torque, as expected on general grounds. This may be simplified to

$$v\frac{\mathrm{d}h}{\mathrm{d}\lambda}=\frac{1}{P}\int R^2A^\phi\,\frac{\mathrm{d}\phi}{\Omega}. \tag{152}$$

Using the expression for A^ϕ, this becomes

$$v\frac{\mathrm{d}h}{\mathrm{d}\lambda}\int J\tilde{\Sigma}\,\mathrm{d}\phi=\partial_\lambda\int JR^2\mathcal{T}^{\lambda\phi}\,\mathrm{d}\phi+\int J\tilde{\Sigma}R^2f_0^\phi\,\mathrm{d}\phi. \tag{153}$$

It remains to determine the evolution of the eccentricity. Here one uses the fact that the operator acting on the unknown $\boldsymbol{v}$ in equations (148) and (149) is singular, and the equations can therefore be solved (in principle) only when $\boldsymbol{B}$ satisfies a certain solvability condition. Eliminate v^λ between equations (148) and (149) to obtain

$$\mathcal{L}v^\phi=\mathcal{R}, \tag{154}$$

where the linear operator $\mathcal{L}$ is defined by

$$\mathcal{L}v^\phi=\Omega\partial_\phi\left[\frac{R_\lambda^2}{R^2}\Omega\partial_\phi(R^2v^\phi)\right]+2\frac{\mathrm{d}h}{\mathrm{d}\lambda}\frac{R_\lambda}{\lambda}\Omega v^\phi, \tag{155}$$

and the right-hand side is

$$\mathcal{R}=\Omega\partial_\phi(R_\lambda^2B^\phi)-\frac{\mathrm{d}h}{\mathrm{d}\lambda}\frac{R_\lambda^2}{R^2}B^\lambda. \tag{156}$$

Now $\mathcal{L}$ is a singular operator because it possesses a null eigenvector

$$w^\phi=\frac{\mathrm{e}^{\mathrm{i}\phi}}{R_\lambda} \tag{157}$$

such that $\mathcal{L}w^\phi=0$. This can be verified by direct substitution, with the help of equations (123) and (125). This null solution corresponds simply to a small redefinition of the eccentricity, $E\mapsto E+\delta E$. The corresponding solvability condition for equation (154) is

$$\int J\tilde{\Sigma}R^2w^\phi\mathcal{R}\,\mathrm{d}\phi=0. \tag{158}$$

Multiplying this by $2\mathrm{i}\lambda/GM$ and taking into account the contributions to $\boldsymbol{B}$ from $\dot{\lambda}$, this can be brought into the form

$$\dot{E}\int J\tilde{\Sigma}\,\mathrm{d}\phi=\frac{2\mathrm{i}\lambda}{GM}\int J\tilde{\Sigma}\frac{R^2\mathrm{e}^{\mathrm{i}\phi}}{R_\lambda}\left[\Omega\partial_\phi(R_\lambda^2A^\phi)-\frac{\mathrm{d}h}{\mathrm{d}\lambda}\frac{R_\lambda^2}{R^2}A^\lambda\right]\mathrm{d}\phi, \tag{159}$$

where the relation

$$\frac{R_\lambda\dot{\lambda}}{R}=\frac{\dot{e}\cos\theta+e\dot{\omega}\sin\theta}{1+e\cos\theta} \tag{160}$$

has been used, and also the integrals

$$\int_0^{2\pi}(1+e\cos\theta)^{-3}\,\mathrm{d}\theta=(2+e^2)\pi(1-e^2)^{-5/2}, \tag{161}$$

$$\int_0^{2\pi}(1+e\cos\theta)^{-3}\cos\theta\,\mathrm{d}\theta=-3e\pi(1-e^2)^{-5/2}, \tag{162}$$

$$\int_0^{2\pi}(1+e\cos\theta)^{-3}\cos^2\theta\,\mathrm{d}\theta=(1+2e^2)\pi(1-e^2)^{-5/2}, \tag{163}$$

which are easily established by the residue theorem. Thus the required equation for the evolution of E has been obtained, but must now be manipulated into a more usable form.

With the help of the relation

$$\frac{\lambda^2R_\lambda^2}{R^4}\partial_\phi\left(\frac{R^2\mathrm{e}^{\mathrm{i}\phi}}{R_\lambda}\right)=\mathrm{i}(\mathrm{e}^{\mathrm{i}\phi}+E-\lambda E'), \tag{164}$$

one obtains

$$\begin{aligned}\dot{E}\int J\tilde{\Sigma}\,\mathrm{d}\phi=&-\frac{\mathrm{i}}{GM}\int J\tilde{\Sigma}\Omega R^2R_\lambda\mathrm{e}^{\mathrm{i}\phi}A^\lambda\,\mathrm{d}\phi\\&+\frac{2}{GM\lambda}\int J\tilde{\Sigma}\Omega(\mathrm{e}^{\mathrm{i}\phi}+E-\lambda E')R^4A^\phi\,\mathrm{d}\phi.\end{aligned} \tag{165}$$

This can be further simplified to

$$h\left(\dot{E}+vE'-\frac{vE}{\lambda}\right)=\frac{1}{P}\int(2R^2A^\phi-\mathrm{i}\lambda R_\lambda A^\lambda)\,\mathrm{e}^{\mathrm{i}\phi}\,\frac{\mathrm{d}\phi}{\Omega}. \tag{166}$$

Taking into account the various contributions to $\boldsymbol{A}$, this becomes

$$\begin{aligned}h\Big(\dot{E}+vE'-\frac{vE}{\lambda}\Big)\int J\tilde{\Sigma}\,\mathrm{d}\phi=&-\frac{3\mathrm{i}}{2}\int J\tilde{\mathcal{I}}\Omega^2\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi\\&-\mathrm{i}\lambda\partial_\lambda\int JR_\lambda\mathcal{T}^{\lambda\lambda}\,\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi+2\partial_\lambda\int JR^2\mathcal{T}^{\lambda\phi}\,\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi\\&-\frac{1}{\lambda}\int JR^2\mathcal{T}^{\lambda\phi}(\mathrm{e}^{\mathrm{i}\phi}+E-\lambda E')\,\mathrm{d}\phi\\&-\mathrm{i}\int JR^2\mathcal{T}^{\phi\phi}\,\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi\\&+\int J\tilde{\Sigma}(2R^2f_0^\phi-\mathrm{i}\lambda R_\lambda f_0^\lambda)\,\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi.\end{aligned} \tag{167}$$

Thus a complete set of one-dimensional equations has been obtained that determine the evolution of the mass, angular momentum and eccentricity vector due to internal stresses and external forcing. It remains to evaluate the equations explicitly for the case of a Maxwellian viscoelastic model with an alpha viscosity.

4.4 Vertical structure and vertical motion

The shear tensor of the orbital motion will be required below. This is defined in general by

$$S^{ab}=\tfrac{1}{2}(\nabla^au^b+\nabla^bu^a), \tag{168}$$

and has the expansion

$$S^{ab}=S_0^{ab}+O(\epsilon). \tag{169}$$

The horizontal components at leading order are

$$S_0^{\lambda\lambda}=\frac{(R^3R_{\lambda\phi}+RR_{\lambda\phi}R_\phi^2+R_\lambda R_\phi^3-RR_\lambda R_\phi R_{\phi\phi})\Omega}{R^3R_\lambda^3}, \tag{170}$$

$$S_0^{\lambda\phi}=\frac{(R^2+R_\phi^2)\partial_\lambda\Omega+(R_\lambda R_{\phi\phi}-R_\phi R_{\lambda\phi})\Omega}{2R^2R_\lambda^2}, \tag{171}$$

$$S_0^{\phi\phi}=-\frac{R_\phi}{R^3R_\lambda}(R\partial_\lambda\Omega+R_\lambda\Omega),\qquad(172)$$

where equation (123) has been used. The orbital contribution to the divergence $\nabla_a u^a$ is

$$\Delta=\frac{1}{J}\partial_\phi(J\Omega)=\left(\frac{GM}{\lambda^3}\right)^{1/2}g_1,\qquad(173)$$

where $g_1(\theta;\lambda,T)$ is a dimensionless quantity, equal to zero for a circular disc (see Appendix A). Note, however, that there is also a vertical shear component $S_0^{zz}=\partial_\zeta u_1^z$ of the same order, which remains to be evaluated. This contributes to both the divergence and the dissipation rate at leading order.

Noting that

$$\int\frac{\mathrm{d}\phi}{\Omega}=P=2\pi(1-e^2)^{-3/2}\left(\frac{\lambda^3}{GM}\right)^{1/2}\qquad(174)$$

is the orbital period, one may write

$$\tilde\Sigma=\Sigma\,g_2,\qquad(175)$$

where $g_2(\theta;\lambda,T)$ is a second dimensionless quantity, equal to unity for a circular disc, and satisfying

$$(1+e\cos\theta)^2\partial_\theta\ln g_2=-g_1.\qquad(176)$$

Further useful quantities are defined according to

$$\lambda\partial_\lambda\Omega=\left(\frac{GM}{\lambda^3}\right)^{1/2}g_3,\qquad(177)$$

$$\partial_\phi\Omega=\left(\frac{GM}{\lambda^3}\right)^{1/2}g_4\qquad(178)$$

and

$$S_0^{ab}=\left(\frac{GM}{\lambda^3}\right)^{1/2}s^{ab},\qquad(179)$$

so that $(g_3,g_4,s^{\lambda\lambda},\lambda s^{\lambda\phi},\lambda^2s^{\phi\phi})$ are dimensionless. These quantities are written out in Appendix A.

The equation of mass conservation (67) at leading order $[O(\epsilon^s)]$ is

$$(\Omega\partial_\phi+u_1^z\partial_\zeta)\rho_0=-\rho_0(\Delta+\partial_\zeta u_1^z).\qquad(180)$$

The energy equation (70) at leading order $[O(\epsilon^{s+2})]$ is

$$\begin{aligned}&\left(\frac{1}{\gamma-1}\right)(\Omega\partial_\phi+u_1^z\partial_\zeta)p_0=-\left(\frac{\gamma}{\gamma-1}\right)p_0(\Delta+\partial_\zeta u_1^z)\\&\quad+T_0^{\lambda\lambda}S_{\lambda\lambda0}+2T_0^{\lambda\phi}S_{\lambda\phi0}+T_0^{\phi\phi}S_{\phi\phi0}+T_0^{zz}\partial_\zeta u_1^z\\&\quad-\partial_\zeta F_0.\end{aligned}\qquad(181)$$

The vertical component of the equation of motion (68) at leading order $[O(\epsilon^{s+1})]$ is

$$\rho_0(\Omega\partial_\phi+u_1^z\partial_\zeta)u_1^z=-\rho_0\Phi_2\zeta-\partial_\zeta p_0+\partial_\zeta T_0^{zz}.\qquad(182)$$

The required components of the stress equation (69) at leading order $[O(\epsilon^{s+2})]$ are

$$\begin{aligned}&T_0^{\lambda\lambda}+\tau\left[(\Omega\partial_\phi+u_1^z\partial_\zeta)T_0^{\lambda\lambda}+2(\Delta+\partial_\zeta u_1^z)T_0^{\lambda\lambda}\right]\\&\quad=2\mu_0S_0^{\lambda\lambda}+(\mu_{\mathrm{b}0}-\tfrac{2}{3}\mu_0)(\Delta+\partial_\zeta u_1^z)g^{\lambda\lambda},\end{aligned}\qquad(183)$$

$$\begin{aligned}&T_0^{\lambda\phi}+\tau\left[(\Omega\partial_\phi+u_1^z\partial_\zeta)T_0^{\lambda\phi}-T_0^{\lambda\lambda}\partial_\lambda\Omega-T_0^{\lambda\phi}\partial_\phi\Omega\right.\\&\quad\left.+2(\Delta+\partial_\zeta u_1^z)T_0^{\lambda\phi}\right]\\&\quad=2\mu_0S_0^{\lambda\phi}+(\mu_{\mathrm{b}0}-\tfrac{2}{3}\mu_0)(\Delta+\partial_\zeta u_1^z)g^{\lambda\phi},\end{aligned}\qquad(184)$$

$$\begin{aligned}&T_0^{\phi\phi}+\tau\left[(\Omega\partial_\phi+u_1^z\partial_\zeta)T_0^{\phi\phi}-2T_0^{\lambda\phi}\partial_\lambda\Omega-2T_0^{\phi\phi}\partial_\phi\Omega\right.\\&\quad\left.+2(\Delta+\partial_\zeta u_1^z)T_0^{\phi\phi}\right]\\&\quad=2\mu_0S_0^{\phi\phi}+(\mu_{\mathrm{b}0}-\tfrac{2}{3}\mu_0)(\Delta+\partial_\zeta u_1^z)g^{\phi\phi},\end{aligned}\qquad(185)$$

$$\begin{aligned}&T_0^{zz}+\tau\left[(\Omega\partial_\phi+u_1^z\partial_\zeta)T_0^{zz}+2\Delta T_0^{zz}\right]\\&\quad=2\mu_0\partial_\zeta u_1^z+(\mu_{\mathrm{b}0}-\tfrac{2}{3}\mu_0)(\Delta+\partial_\zeta u_1^z).\end{aligned}\qquad(186)$$

The constitutive relations at leading order are

$$F_0=-\frac{16\sigma T_0^{3-y}}{3C_\kappa\rho_0^{1+x}}\partial_\zeta T_0,\qquad(187)$$

$$p_0=\frac{k\rho_0T_0}{\mu_{\mathrm{m}}m_{\mathrm{H}}},\qquad(188)$$

$$\mu_0=\alpha p_0\left(\frac{GM}{\lambda^3}\right)^{-1/2},\qquad\mu_{\mathrm{b}0}=\alpha_{\mathrm{b}}p_0\left(\frac{GM}{\lambda^3}\right)^{-1/2}.\qquad(189)$$

The solution of equations (180)–(189) is found by a *non-linear separation of variables*, a similar method to that used for warped discs (Ogilvie 2000). As an intermediate step, one proposes that the solution should satisfy the *generalized vertical equilibrium* relations

$$\begin{aligned}\frac{\partial p_0}{\partial\zeta}&=-f_2\,\rho_0\left(\frac{GM}{\lambda^3}\right)\zeta,&(190)\\\frac{\partial F_0}{\partial\zeta}&=f_1\,\frac{9\alpha}{4}p_0\left(\frac{GM}{\lambda^3}\right)^{1/2},&(191)\end{aligned}$$

in addition to equations (187)–(189). Here $f_1(\theta;\lambda,T)$ and $f_2(\theta;\lambda,T)$ are dimensionless functions to be determined, and which are equal to unity for a circular disc. Physically, f_1 differs from unity in an eccentric disc because of the enhanced dissipation of energy and because of compressive heating and cooling (equation 181). Similarly, f_2 reflects changes in the usual hydrostatic vertical equilibrium resulting from the vertical velocity, the vertical viscous stress and the variation of the vertical oscillation frequency around the orbit (equation 182). Following Ogilvie (2000), one solves these equations by reducing them to a standard dimensionless form. One identifies a natural physical unit for the thickness of the disc,

$$\begin{aligned}U_H=&\left(\frac{9\alpha}{4}\right)^{1/(6+x-2y)}\Sigma^{(2+x)/(6+x-2y)}\left(\frac{GM}{\lambda^3}\right)^{-(5-2y)/2(6+x-2y)}\\&\times\left(\frac{\mu_{\mathrm{m}}m_{\mathrm{H}}}{k}\right)^{-(4-y)/(6+x-2y)}\left(\frac{16\sigma}{3C_\kappa}\right)^{-1/(6+x-2y)},\end{aligned}\qquad(192)$$

and for the other variables according to

$$U_\rho=\Sigma U_H^{-1},\qquad(193)$$

$$U_p=\left(\frac{GM}{\lambda^3}\right)U_H^2U_\rho,\qquad(194)$$

$$U_T=\left(\frac{GM}{\lambda^3}\right)\left(\frac{\mu_{\mathrm{m}}m_{\mathrm{H}}}{k}\right)U_H^2,\qquad(195)$$

$$U_F=\left(\frac{9\alpha}{4}\right)\left(\frac{GM}{\lambda^3}\right)^{1/2}U_HU_p.\qquad(196)$$

The solution of the generalized vertical equilibrium equations is then

$$\begin{aligned}\zeta=&\zeta_*\,f_1^{1/(6+x-2y)}f_2^{-(3-y)/(6+x-2y)}\\&\times g_2^{(2+x)/(6+x-2y)}U_H,\end{aligned}\qquad(197)$$

$$\rho_0=\rho_*(\zeta_*)\,f_1^{-1/(6+x-2y)}\,f_2^{(3-y)/(6+x-2y)}\times g_2^{2(2-y)/(6+x-2y)}U_\rho, \qquad (198)$$

$$p_0=p_*(\zeta_*)\,f_1^{1/(6+x-2y)}\,f_2^{(3+x-y)/(6+x-2y)}\times g_2^{2(4+x-y)/(6+x-2y)}U_p, \qquad (199)$$

$$T_0=T_*(\zeta_*)\,f_1^{2/(6+x-2y)}\,f_2^{x/(6+x-2y)}\times g_2^{2(2+x)/(6+x-2y)}U_T, \qquad (200)$$

$$F_0=F_*(\zeta_*)\,f_1^{(8+x-2y)/(6+x-2y)}\,f_2^{x/(6+x-2y)}\times g_2^{(10+3x-2y)/(6+x-2y)}U_F, \qquad (201)$$

where the starred variables satisfy the dimensionless ODEs

$$\frac{\mathrm{d}p_*}{\mathrm{d}\zeta_*}=-\rho_*\zeta_*, \qquad (202)$$

$$\frac{\mathrm{d}F_*}{\mathrm{d}\zeta_*}=p_*, \qquad (203)$$

$$\frac{\mathrm{d}T_*}{\mathrm{d}\zeta_*}=-\rho_*^{1+x}T_*^{-3+y}F_*, \qquad (204)$$

$$p_*=\rho_*T_*, \qquad (205)$$

subject to the boundary conditions

$$F_*(0)=\rho_*(\zeta_{s*})=T_*(\zeta_{s*})=0 \qquad (206)$$

and the normalization of surface density,

$$\int_{-\zeta_{s*}}^{\zeta_{s*}}\rho_*\,\mathrm{d}\zeta_*=1. \qquad (207)$$

Here ζ_{s*} is the dimensionless height of the upper surface of the disc. Also required is the dimensionless second vertical moment of the density,

$$\mathcal{I}_*=\int_{-\zeta_{s*}}^{\zeta_{s*}}\rho_*\zeta_*^2\,\mathrm{d}\zeta_*. \qquad (208)$$

The solution of these dimensionless ODEs depends only on the opacity law and is easily obtained numerically (Ogilvie 2000).¶ For Thomson opacity one finds $\zeta_{s*}\approx 2.383$ and $\mathcal{I}_*=0.6777$. For Kramers opacity one finds $\zeta_{s*}\approx 2.543$ and $\mathcal{I}_*=0.7094$.

One further proposes that the vertical velocity is of the form

$$u_1^z=f_3\left(\frac{GM}{\lambda^3}\right)^{1/2}\zeta, \qquad (209)$$

where $f_3(\theta;\lambda,T)$ is a third dimensionless function, equal to zero for a circular disc, and that the stress components at leading order are of the form

$$T_0^{ab}=t^{ab}(\theta;\lambda,T)\,p_0, \qquad (210)$$

so that $(t^{\lambda\lambda},\lambda t^{\lambda\phi},\lambda^2t^{\phi\phi},t^{zz})$ are dimensionless coefficients.

When these tentative solutions are substituted into equations (180)–(186) one obtains a number of dimensionless ODEs in θ, which must be satisfied if the solution is to be valid. From equation (180) one obtains

$$(1+e\cos\theta)^2\left[-\partial_\theta\ln f_1+(3-y)\partial_\theta\ln f_2\right]=-(6+x-2y)f_3-(2+x)g_1. \qquad (211)$$

Similarly, from equation (181) one obtains

$$(1+e\cos\theta)^2\left(\frac{1}{\gamma-1}\right)\partial_\theta\ln f_2=-\left(\frac{\gamma+1}{\gamma-1}\right)f_3-g_1+t^{\lambda\lambda}s_{\lambda\lambda}+2t^{\lambda\phi}s_{\lambda\phi}+t^{\phi\phi}s_{\phi\phi}+t^{zz}f_3-\frac{9}{4}\alpha f_1. \qquad (212)$$

From equation (182) one obtains

$$(1+e\cos\theta)^2\partial_\theta f_3=-f_3^2-(1+e\cos\theta)^3+f_2(1-t^{zz}). \qquad (213)$$

Finally, from equations (183)–(186) one obtains

$$t^{\lambda\lambda}+\mathrm{We}\left[(1+e\cos\theta)^2(\partial_\theta t^{\lambda\lambda}+t^{\lambda\lambda}\partial_\theta\ln f_2)+(3f_3+g_1)t^{\lambda\lambda}\right]=2\alpha s^{\lambda\lambda}+(\alpha_\mathrm{b}-\tfrac{2}{3}\alpha)(g_1+f_3)g^{\lambda\lambda}, \qquad (214)$$

$$\lambda t^{\lambda\phi}+\mathrm{We}\left[(1+e\cos\theta)^2(\partial_\theta(\lambda t^{\lambda\phi})+\lambda t^{\lambda\phi}\partial_\theta\ln f_2)+(3f_3+g_1)\lambda t^{\lambda\phi}-g_3t^{\lambda\lambda}-g_4\lambda t^{\lambda\phi}\right]=2\alpha\lambda s^{\lambda\phi}+(\alpha_\mathrm{b}-\tfrac{2}{3}\alpha)(g_1+f_3)\lambda g^{\lambda\phi}, \qquad (215)$$

$$\lambda^2t^{\phi\phi}+\mathrm{We}\left[(1+e\cos\theta)^2(\partial_\theta(\lambda^2t^{\phi\phi})+\lambda^2t^{\phi\phi}\partial_\theta\ln f_2)+(3f_3+g_1)\lambda^2t^{\phi\phi}-2g_3\lambda t^{\lambda\phi}-2g_4\lambda^2t^{\phi\phi}\right]=2\alpha\lambda^2s^{\phi\phi}+(\alpha_\mathrm{b}-\tfrac{2}{3}\alpha)(g_1+f_3)\lambda^2g^{\phi\phi}, \qquad (216)$$

$$t^{zz}+\mathrm{We}\left[(1+e\cos\theta)^2(\partial_\theta t^{zz}+t^{zz}\partial_\theta\ln f_2)+(f_3+g_1)t^{zz}\right]=2\alpha f_3+(\alpha_\mathrm{b}-\tfrac{2}{3}\alpha)(g_1+f_3). \qquad (217)$$

These ODEs should be solved for the functions $(f_1,f_2,f_3,t^{\lambda\lambda},\lambda t^{\lambda\phi},\lambda^2t^{\phi\phi},t^{zz})$ subject to periodic boundary conditions $f_1(2\pi;\lambda,T)=f_1(0;\lambda,T)$, etc. Note that, in the case $\mathrm{We}=0$, the equations for t^{ab} become algebraic, reducing the problem to third order. Note also that, in the limit of a circular disc ($e=0$, $g_1=0$, $g_2=1$, $g_3=-\frac{3}{2}$, $g_4=0$, $g^{\lambda\lambda}=1$, $\lambda g^{\lambda\phi}=0$, $\lambda^2g^{\phi\phi}=1$, $s_{\lambda\lambda}=0$, $\lambda^{-1}s_{\lambda\phi}=-\frac{3}{4}$, $\lambda^{-2}s_{\phi\phi}=0$), the solution is $f_1=1$, $f_2=1$, $f_3=0$, $t^{\lambda\lambda}=0$, $\lambda t^{\lambda\phi}=-\frac{3}{2}\alpha$, $\lambda^2t^{\phi\phi}=\frac{9}{2}\mathrm{We}\,\alpha$, $t^{zz}=0$.

4.5 Evaluation of the stress integrals

The evolutionary equations may be written in the form

$$\dot\Sigma+\frac{1}{\lambda}\partial_\lambda(\lambda v\Sigma)=0, \qquad (218)$$

$$\Sigma v\frac{\mathrm{d}h}{\mathrm{d}\lambda}=\frac{1}{\lambda}\partial_\lambda\left(Q_1\mathcal{I}\frac{GM}{\lambda}\right)+\frac{\Sigma}{P}\int R^2f_0^\phi\,\frac{\mathrm{d}\phi}{\Omega}, \qquad (219)$$

$$\Sigma h\left(\dot E+vE'-\frac{vE}{\lambda}\right)=\partial_\lambda\left(Z_1\mathcal{I}\frac{GM}{\lambda^2}\right)+Z_2\mathcal{I}\frac{GM}{\lambda^3}+\frac{\Sigma}{P}\int(2R^2f_0^\phi-\mathrm{i}\lambda R_\lambda f_0^\lambda)\,\mathrm{e}^{\mathrm{i}\phi}\,\frac{\mathrm{d}\phi}{\Omega}, \qquad (220)$$

where Q_1 is a dimensionless real coefficient defined by

$$\int JR^2\mathcal{T}^{\lambda\phi}\,\mathrm{d}\phi=Q_1\frac{GM}{\lambda^2}\int J\tilde{\mathcal{I}}\,\mathrm{d}\phi, \qquad (221)$$

and Z_1 and Z_2 are dimensionless complex coefficients defined by

¶ It is assumed here that the disc is highly optically thick so that 'zero boundary conditions' are adequate. The corrections for finite optical depth are described by Ogilvie (2000).

$$\frac{1}{\lambda}\int J(2R^2\mathcal{T}^{\lambda\phi}-\mathrm{i}\lambda R_\lambda\mathcal{T}^{\lambda\lambda})\,\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi=Z_1\frac{GM}{\lambda^3}\int J\tilde{\mathcal{I}}\,\mathrm{d}\phi,\tag{222}$$

$$-\frac{3\mathrm{i}}{2}\int J\tilde{\mathcal{I}}\Omega^2\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi+\frac{1}{\lambda}\int JR^2\mathcal{T}^{\lambda\phi}(\mathrm{e}^{\mathrm{i}\phi}-E+\lambda E')\,\mathrm{d}\phi$$
$$-\mathrm{i}\int JR^2\mathcal{T}^{\phi\phi}\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi=Z_2\frac{GM}{\lambda^3}\int J\tilde{\mathcal{I}}\,\mathrm{d}\phi.\tag{223}$$

To evaluate the stress integrals, first note the relation

$$\int p_0\,\mathrm{d}\zeta=f_2\left(\frac{GM}{\lambda^3}\right)\tilde{\mathcal{I}},\tag{224}$$

which follows from equation (190) after an integration by parts. Then

$$\mathcal{T}^{ab}=(t^{ab}-g^{ab})f_2\left(\frac{GM}{\lambda^3}\right)\tilde{\mathcal{I}}.\tag{225}$$

Define the dimensionless function $f_4(\theta;\lambda,T)$ by

$$J\tilde{\mathcal{I}}=f_4\lambda\mathcal{I},\tag{226}$$

so that

$$f_4=\frac{f_5}{\frac{1}{2\pi}\int f_5\,\mathrm{d}\theta},\tag{227}$$

where

$$f_5=(1+e\cos\theta)^{-2}f_1^{2/(6+x-2y)}f_2^{-2(3-y)/(6+x-2y)}$$
$$\times g_2^{2(2+x)/(6+x-2y)}.\tag{228}$$

Then one finds

$$Q_1=\frac{1}{2\pi}\int f_2f_4(1+e\cos\theta)^{-2}\lambda(t^{\lambda\phi}-g^{\lambda\phi})\,\mathrm{d}\theta,\tag{229}$$

$$Z_1=\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}\omega}\int f_2f_4\left[2(1+e\cos\theta)^{-2}\lambda(t^{\lambda\phi}-g^{\lambda\phi})\right.$$
$$\left.-\mathrm{i}R_\lambda(t^{\lambda\lambda}-g^{\lambda\lambda})\right]\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta,\tag{230}$$

$$Z_2=\frac{1}{2\pi}(\lambda E'-E)\int f_2f_4(1+e\cos\theta)^{-2}\lambda(t^{\lambda\phi}-g^{\lambda\phi})\,\mathrm{d}\theta$$
$$+\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}\omega}\int\left\{-\frac{3\mathrm{i}}{2}f_4(1+e\cos\theta)^4+f_2f_4(1+e\cos\theta)^{-2}\right.$$
$$\left.\times\left[\lambda(t^{\lambda\phi}-g^{\lambda\phi})-\mathrm{i}\lambda^2(t^{\phi\phi}-g^{\phi\phi})\right]\right\}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta.\tag{231}$$

Finally, a relation is required between Σ and $\mathcal{I}$. One finds

$$\mathcal{I}=Q_2C_{\mathcal{I}}\alpha^{2/(6+x-2y)}\left(\frac{GM}{\lambda^3}\right)^{-(5-2y)/(6+x-2y)}$$
$$\times\Sigma^{(10+3x-2y)/(6+x-2y)},\tag{232}$$

where

$$C_{\mathcal{I}}=\left(\frac{9}{4}\right)^{2/(6+x-2y)}\mathcal{I}_*\left(\frac{\mu_\mathrm{m}m_\mathrm{H}}{k}\right)^{-2(4-y)/(6+x-2y)}$$
$$\times\left(\frac{16\sigma}{3C_\kappa}\right)^{-2/(6+x-2y)}\tag{233}$$

is a constant and Q_2 is a dimensionless coefficient defined by

$$Q_2=\frac{1}{2\pi}(1-e^2)^{3/2}\int f_5\,\mathrm{d}\theta,\tag{234}$$

and equal to unity for a circular disc. The dimensionless coefficients can all be evaluated numerically from the solution of the dimensionless ODEs in Section 4.4. For given values of the model parameters $(\alpha,\alpha_\mathrm{b},\mathrm{We},\gamma,x,y)$, the quantities $(Q_1,Q_2,Z_1/\mathrm{e}^{\mathrm{i}\omega},Z_2/\mathrm{e}^{\mathrm{i}\omega})$ depend only on the dimensionless dynamical variables $(e,\lambda e',\lambda e\omega')$.

4.6 Relation to the Gauss perturbation equations

It may now be shown how the evolutionary equations for a continuous disc relate to those derived in Section 1.2 for a test body. The relation between the ordinary cylindrical polar components $(f_{\hat{R}},f_{\hat{\phi}})$ of the perturbing force and the contravariant orbital components (f^λ,f^ϕ) is

$$f_{\hat{R}}=R_\lambda f^\lambda+R_\phi f^\phi,\tag{235}$$

$$f_{\hat{\phi}}=Rf^\phi.\tag{236}$$

Thus equation (6) for the test body becomes

$$\frac{\mathrm{d}h}{\mathrm{d}t}=v\frac{\mathrm{d}h}{\mathrm{d}\lambda}=R^2f^\phi,\tag{237}$$

where $v=\mathrm{d}\lambda/\mathrm{d}t$. This is equivalent to equation (152) for the disc, except that the disc equation involves a time-average over the orbit, and the perturbing force includes contributions from internal stresses.

Similarly, equation (7) for the test body may be (after some algebra) brought into the form

$$h\left(\frac{\mathrm{d}E}{\mathrm{d}t}-\frac{vE}{\lambda}\right)=(2R^2f^\phi-\mathrm{i}\lambda R_\lambda f^\lambda)\,\mathrm{e}^{\mathrm{i}\phi},\tag{238}$$

which is consistent, in the same sense, with equation (166).

4.7 Relation to standard accretion disc theory

In the absence of an external torque, equations (218) and (219) may be combined into a single evolutionary equation for the surface density,

$$\dot{\Sigma}=\frac{3}{\lambda}\partial_\lambda\left\{\lambda^{1/2}\partial_\lambda\left[\lambda^{1/2}\left(-\frac{2Q_1}{3}\right)\mathcal{I}\left(\frac{GM}{\lambda^3}\right)^{1/2}\right]\right\}.\tag{239}$$

In the limit of a circular disc, λ reduces to R and one finds that $Q_1\to-3\alpha/2$. One then obtains the standard diffusion equation for the surface density of a circular accretion disc (e.g. Pringle 1981),

$$\dot{\Sigma}=\frac{3}{R}\partial_R\left[R^{1/2}\partial_R\left(R^{1/2}\nu\Sigma\right)\right],\tag{240}$$

where $\nu=\alpha\Omega(\mathcal{I}/\Sigma)$ is the vertically averaged kinematic viscosity. In an eccentric disc, therefore, the surface density diffuses viscously in much the same way as in a circular disc, except that it is best considered as diffusing in the space of semi-latus rectum λ (which is equivalent to specific angular momentum). Non-linear couplings arise, however, because the coefficient Q_1 depends on the local eccentricity distribution, specifically on e, $\lambda e'$ and $\lambda e\omega'$. The elliptical distortion of the orbits modifies the stresses that leads to accretion. The eccentricity obeys its own equation (220), which has the character of a complex, non-linear advection-diffusion equation.

5 LINEAR THEORY

In general the dimensionless ODEs must be solved numerically and the Q- and Z-coefficients determined from the solution. However, for small eccentricities the equations can be solved by expanding in powers of e. In doing so it is assumed that $\lambda E'$ is $O(e)$, i.e. that the length-scale on which e or ω varies is comparable to the radius of the disc.

One then finds

$$Q_1 = -\frac{3}{2}\alpha + O(e^2), \tag{241}$$

$$Q_2 = 1 + O(e^2), \tag{242}$$

$$Z_1 = c_1 E + c_2 \lambda E' + O(e^3), \tag{243}$$

$$Z_2 = c_3 E + c_4 \lambda E' + O(e^3). \tag{244}$$

The complex coefficients c_i depend only on the dimensionless parameters $(\alpha, \alpha_\mathrm{b}, \mathrm{We}, \gamma, x, y)$, and can be obtained by a straightforward algebraic calculation. Such a calculation is not presented here because the resulting expressions are too complicated to write down, and it is no more difficult to evaluate the dimensionless coefficients numerically in the non-linear case. The only simple, general statement that can be made is that the c_i become purely imaginary in the inviscid limit $\alpha, \alpha_\mathrm{b} \to 0$.

If the higher-order terms are neglected, and in the absence of external forces, one obtains a complex linear diffusion-type equation for the eccentricity, together with the standard evolutionary equation for the surface density, which is decoupled from the eccentricity. This is similar to the situation in Section 2 except that three-dimensional and radiative effects have been included. Of greatest interest is the coefficient c_2, because the small-scale instability discussed in Section 2 exists if $\mathrm{Re}(c_2) < 0$.

For the purposes of illustration, suppose that $\alpha = 0.1$ and $\gamma = 5/3$. In a purely viscous disc with $\mathrm{We} = 0$, it is then found that stability requires a bulk viscosity $\alpha_\mathrm{b} > 0.350$ in the case of Thomson opacity, or $\alpha_\mathrm{b} > 0.176$ in the case of Kramers opacity. It is by no means clear that such values are realistic. However, the instability is easily quenched when a non-zero relaxation time is introduced. Even when $\alpha_\mathrm{b} = 0$, the instability is absent for $0.398 < \mathrm{We} < 2.294$ (Thomson) or $0.333 < \mathrm{We} < 2.467$ (Kramers). This contrasts with the oversimple model of Section 2, which generally predicted narrower stability intervals and suggested that the instability persisted for all We when $\mu_\mathrm{b} = 0$. These results emphasize the importance of including three-dimensional effects and radiative damping when treating eccentric discs.

One of the features of the linear analysis is that shows that vertical motion is always present in an eccentric disc. To understand this, consider a simplified example in which the gas is isothermal so that the pressure is proportional to the density. Suppose further that the eccentricity is instantaneously uniform. In the absence of vertical motion the velocity divergence is zero and the density and pressure must then be constant along any orbit, for any value of z. However, this is incompatible with hydrostatic vertical equilibrium because the vertical gravitational acceleration varies around any eccentric orbit. Of course, this effect is missing in two-dimensional treatments of eccentric discs.

6 AN ILLUSTRATIVE NON-LINEAR CALCULATION

The practical nature of the non-linear evolutionary equations is now demonstrated by a simple illustrative calculation. In any detailed application, careful consideration must be given to the formulation of appropriate boundary conditions and to any sources of mass and/or eccentricity to be added to the equations. At a free boundary of the disc, where $\Sigma \to 0$, equation (220) generally has a singular point and one must select the regular solution. If the disc is terminated by an external agency, however, a different boundary condition on the eccentricity may apply. These issues should be addressed in future work, within the context of specific applications. For the present purposes, it is convenient to study a simple test problem with idealized boundary conditions.

The parameters adopted are $\alpha = 0.1$, $\alpha_\mathrm{b} = 0$, $\mathrm{We} = 0.5$, $\gamma = 5/3$, $x = 1$ and $y = -7/2$, appropriate to a disc in which Kramers opacity is dominant. (The corresponding linear-theory coefficients are $c_1 = 0.075 + 0.972\mathrm{i}$, $c_2 = 0.030 + 0.631\mathrm{i}$, $c_3 = -0.228 + 0.955\mathrm{i}$ and $c_4 = -0.198 - 0.222\mathrm{i}$. Therefore the short-wavelength instability is absent.) For this opacity law there exists a self-similar solution with $\Sigma \propto \lambda^{-3/4}$ and $E = 0$, representing a steady, circular accretion disc having an arbitrarily small inner radius. According to the theories of Syer & Clarke (1992, 1993) and Lyubarskij et al. (1994), if such a disc is made uniformly eccentric, with all the orbits aligned, it should remain so. In order to contrast the results of the present theory with these earlier calculations, the non-linear evolutionary equations (218)–(220) are solved starting from an initial condition with surface density $\Sigma \propto \lambda^{-3/4}$ and uniform eccentricity $E = 0.3$. The equations are solved in a finite domain, $1 < \lambda < 100$, with boundary conditions $\partial \ln \Sigma / \partial \ln \lambda = -3/4$ and $\partial_\lambda E = 0$ at the inner and outer edges. These illustrative boundary conditions are chosen to be relatively neutral while being compatible with the notional solution proposed by Syer & Clarke and Lyubarskij et al. No external forcing is applied.

The equations are first discretized in space by representing the variables Σ and E on a set of 200 logarithmically spaced orbits. The derivatives are represented by simple, centred finite differences and the scheme conserves mass exactly. The resulting set of temporal ODEs is solved using a fifth-order Runge-Kutta method with adaptive step-size. The time-step adjusts automatically to ensure accuracy and stability of the integration.

Before the start of the run, the dimensionless coefficients $(Q_1, Q_2, Z_1/\mathrm{e}^{\mathrm{i}\omega}, Z_2/\mathrm{e}^{\mathrm{i}\omega})$ are evaluated on a rectangular grid in the space $(e, \lambda e', \lambda e \omega')$ by solving the dimensionless ODEs of Section 4.4 in a once-for-all calculation for the chosen parameter values. During the run, the coefficients are then evaluated rapidly by trilinear interpolation on the grid.

Fig. 2 shows the evolution of e and ω over a time interval $100 t_\nu$, where $t_\nu = \lambda^2/\nu$ is the viscous time-scale at the inner boundary, and $\nu = \alpha\Omega(\mathcal{I}/\Sigma)$ is the vertically averaged kinematic viscosity in a circular disc. During this period the surface density exhibits negligible evolution. However, there is a rapid twisting of the disc because of differential precession caused by the radial pressure gradient, an effect explained in Section 2.2. After a transient phase, the eccen-

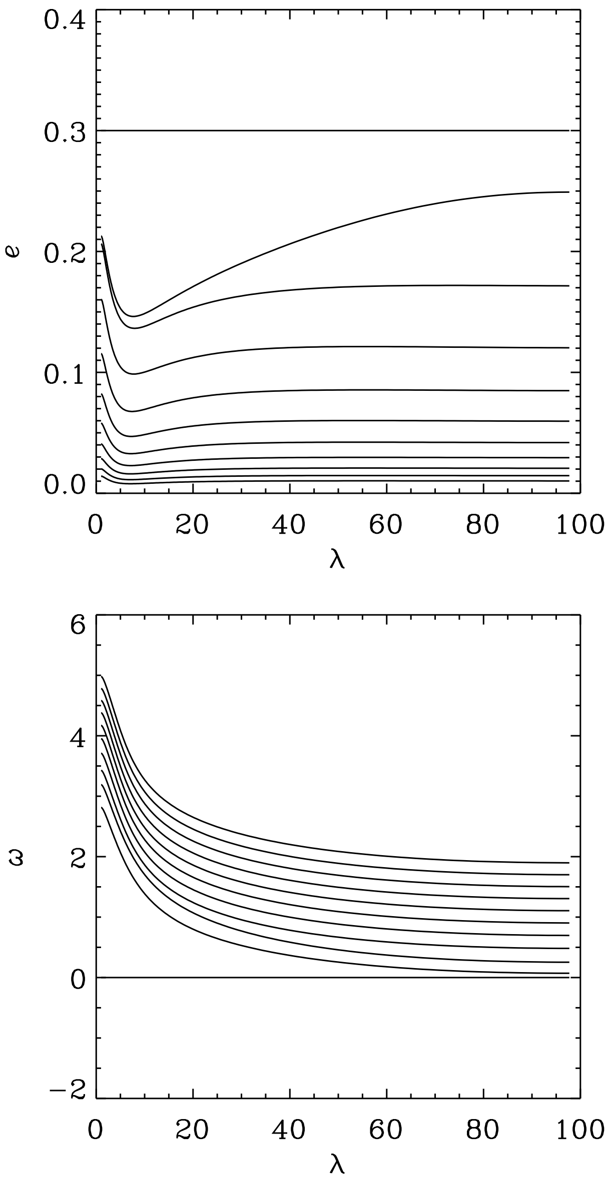

Figure 2. Top: Evolution of the eccentricity in an initially uniformly eccentric disc. The curves are ordered from top to bottom, and are separated by time intervals of $10t_\nu$. Bottom: Evolution of the longitude of periastron (radians). The curves are ordered from bottom to top.

tricity settles into a twisted mode that decays exponentially in time and precesses slowly in a prograde sense.

Owing to the monotonic decay of the eccentricity, non-linear terms are active only in the earliest stages of the evolution. However, non-linear effects will be critical in other applications, such as determining the outcome of the eccentric instability in superhump binaries.

7 DISCUSSION

In this paper a comprehensive theory of eccentric accretion discs has been presented. Starting from the basic fluid-dynamical equations in three dimensions, I have derived the fundamental set of one-dimensional equations that describe how the mass, angular momentum and eccentricity vector of a thin disc evolve as a result of internal stresses and external forcing (equations 218–220). The analysis is asymptotically exact in the limit of a thin disc, and allows for slowly varying eccentricities of arbitrary magnitude.

These equations are generally valid and therefore of fundamental interest. They are the equivalent of the Gauss perturbation equations for a continuous disc. Previously, Lyubarskij et al. (1994) succeeded in deriving a related set of equations for an eccentric disc by considering the conservation of mass, angular momentum and energy. However, their analysis is restricted to the case in which the ellipses are all aligned and do not precess. Their method works in this case because a knowledge of the angular momentum and energy of an orbiting body is sufficient to determine its semi-latus rectum and eccentricity, but not its longitude of periastron. A closed system of equations is obtained only if the ellipses are artificially constrained not to precess. In reality, such precession is inevitable and the evolution of the longitude of periastron must be determined from a full analysis of the horizontal components of the equation of motion. This leads to an equation for the eccentricity vector, or complex eccentricity, which is not in conservative form.

The second achievement of this paper is the explicit development of the equations in the case of a specific stress model which, it is hoped, gives a fair representation of the turbulent stress in an accretion disc. To obtain the coefficients in the evolutionary equations requires a solution of the non-linear PDEs that govern the azimuthal and vertical structure of the disc. It also requires an understanding of the relation between the turbulent stress tensor and the velocity gradient tensor. The simplest plausible relation, adopted in almost all theoretical work on accretion discs, is an effective viscosity model in which an instantaneous linear relation is assumed, and the equation of motion therefore reduces to the Navier-Stokes equation. In this paper I have introduced a Maxwellian viscoelastic model of the turbulent stress in an accretion disc. This generalizes the conventional alpha viscosity model to account for the non-zero relaxation time of the turbulence, and is physically motivated by a consideration of the nature of MHD turbulence. The PDEs governing the azimuthal and vertical structure of the disc, including the effects of vertical motion, dissipation of energy and radiative transport, have been reduced exactly to a set of dimensionless ODEs which can be solved numerically to high accuracy to yield the coefficients required for the evolutionary equations. This shows that the technique of non-linear separation of variables, applied first to warped discs (Ogilvie 1999, 2000), is not restricted to purely viscous models but can incorporate improved representations of the stress as our understanding of magnetorotational turbulence develops.

It has been confirmed that circular discs are usually viscously unstable to short-wavelength eccentric perturbations, as found by Lyubarskij et al. (1994), if the conventional alpha viscosity model is adopted. It has been noted that the instability is essentially the same as the viscous overstability of axisymmetric modes discovered by Kato (1978). The instability can be suppressed by introducing a sufficient effective bulk viscosity, although the values required may not be realistic. More plausibly, the instability can be suppressed by allowing for the non-zero relaxation time of the turbulence,

even if the bulk viscosity is zero. It has then been shown that an initially uniformly eccentric disc does not retain its eccentricity over many viscous time-scales, as had been suggested by Syer & Clarke (1992, 1993) and Lyubarskij et al. (1994). These earlier works neglected the differential precession caused by slightly non-Keplerian rotation resulting from the radial pressure gradient. This leads to twisting of the disc, followed by viscous decay of the eccentricity.

The theory presented here goes considerably beyond previous analytical treatments of eccentric discs. It also provides a practical numerical scheme that involves only one-dimensional equations and from which the fast orbital time-scale has been eliminated. This scheme is to be preferred in many circumstances to a direct numerical simulation of the fluid-dynamical equations. Almost all direct simulations to date attempt to represent the Navier-Stokes equation (with or without explicit viscosity) in two dimensions. The present analysis shows that a two-dimensional treatment of eccentric discs may capture many of the correct qualitative features but cannot be trusted in detail. Vertical motion is always present in eccentric discs and radiative damping can influence the evolution of eccentricity. Furthermore, the Navier-Stokes equation does not take into account the relaxation time of the turbulence, which can be of great importance in this context.

Nevertheless, it would be valuable to make detailed comparisons between the present theory and direct simulations (preferably three-dimensional). The present theory cannot be applied reliably to thick discs, nor to situations in which the eccentricity varies rapidly in space (i.e. on a length-scale comparable to the thickness of the disc) or in time (i.e. on a time-scale comparable to the orbital period). Mean-motion resonances are therefore excluded from the analysis, which is secular in the sense of celestial mechanics. Nevertheless, the effect of mean-motion resonances could be included in the evolutionary equations by adding appropriate localized source terms for angular momentum and eccentricity.

The theory developed in this paper has much in common with the theory of warped accretion discs (e.g. Pringle 1992; Ogilvie 1999, 2000). One distinction, noted above, is that the equations for an eccentric disc are not all in conservative form. Another difference is that the theory of warped discs is complicated by a resonance caused by the coincidence of the orbital and epicyclic frequencies in a Keplerian disc. As a result, the behaviour is qualitatively different depending on the relative magnitudes of α and H/R (Papaloizou & Lin 1995). Fortunately, no such complication arises in the case of eccentric discs. A consistent asymptotic expansion of the fluid-dynamical equations is possible for any value of α, and the fractional error in the asymptotic approximation, $O((H/R)^2)$, is very small in most applications.

The evolutionary equations should be useful in many applications, including understanding the eccentric planet-disc interaction and testing theories of quasi-periodic oscillations in X-ray binaries. In future work the rate of change of the complex eccentricity caused by external forcing should be evaluated explicitly in the non-linear case for tidal forcing by a companion object on a circular or eccentric orbit, and also for Einstein precession near a black hole.

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APPENDIX A: GEOMETRICAL QUANTITIES

The following dimensionless quantities are required for evaluation of the stress coefficients. These expressions generalize those of Lyubarskij et al. (1994) to allow for precession of the orbits, and also correct a number of errors in that paper. In the following, c and s denote $\cos\theta$ and $\sin\theta$, respectively.

Rate of change of radius with semi-latus rectum:

$$R_\lambda = \frac{1+(e-\lambda e')c-\lambda e\omega' s}{(1+ec)^2}. \tag{A1}$$

Inverse metric coefficients:

$$g^{\lambda\lambda} = \frac{(1+ec)^2(1+2ec+e^2)}{[1+(e-\lambda e')c-\lambda e\omega' s]^2}, \tag{A2}$$

$$\lambda g^{\lambda\phi} = -\frac{(1+ec)^2 es}{1+(e-\lambda e')c-\lambda e\omega' s}, \tag{A3}$$

$$\lambda^2 g^{\phi\phi} = (1+ec)^2. \tag{A4}$$

Orbital contribution to the velocity divergence:

$$g_1 = \frac{(1+ec)(\lambda e' s-\lambda e\omega' c-\lambda e^2\omega')}{1+(e-\lambda e')c-\lambda e\omega' s}. \tag{A5}$$

Variation of the surface density around the orbit:

$$g_2 = \frac{(1-e^2)^{3/2}(1+ec)}{1+(e-\lambda e')c-\lambda e\omega' s}. \tag{A6}$$

Derivatives of the angular velocity:

$$g_3 = -\frac{3}{2}(1+ec)^2 + 2(\lambda e' c+\lambda e\omega' s)(1+ec), \tag{A7}$$

$$g_4 = -2es(1+ec). \tag{A8}$$

Orbital contribution to the shear tensor:

$$\begin{aligned} s^{\lambda\lambda} = {} & \frac{(1+ec)^3}{[1+(e-\lambda e')c-\lambda e\omega' s]^3} \\ & \times \left\{(1+ec)^2 es + \lambda e'(1+ec+e^2s^2)s \right. \\ & \left. -\lambda e\omega'\left[c+e(2+c^2)+e^2(4-c^2)c+e^3\right]\right\}, \end{aligned} \tag{A9}$$

$$\begin{aligned} \lambda s^{\lambda\phi} = {} & \frac{(1+ec)^3}{[1+(e-\lambda e')c-\lambda e\omega' s]^2} \\ & \times \left\{-\frac{3}{4}-\frac{7}{4}ec-\frac{1}{4}e^2(1+4c^2)-\frac{1}{4}e^3c \right. \\ & +\lambda e'\left[c-\frac{1}{2}e(1-4c^2)+\frac{1}{2}e^2c\right] \\ & \left. +\lambda e\omega'(1+2ec+e^2)s\right\}, \end{aligned} \tag{A10}$$

$$\begin{aligned} \lambda^2 s^{\phi\phi} = {} & \frac{(1+ec)^3 es}{1+(e-\lambda e')c-\lambda e\omega' s} \\ & \times \left[\frac{1}{2}(1+ec)-\lambda e' c-\lambda e\omega' s\right]. \end{aligned} \tag{A11}$$

Covariant components:

$$\begin{aligned} s_{\lambda\lambda} = {} & \frac{1}{(1+ec)^2}\left\{-\frac{1}{2}e(1+ec)s+\lambda e'\left(1+\frac{3}{2}ec\right)s \right. \\ & -\lambda e\omega'\left[c-\frac{1}{2}e(1-3c^2)\right] \\ & \left. -\lambda^2 e'^2 cs-\lambda^2 ee'\omega'(1-2c^2)+\lambda^2e^2\omega'^2 cs\right\}, \end{aligned} \tag{A12}$$

$$\begin{aligned} \lambda^{-1}s_{\lambda\phi} = {} & \frac{1}{(1+ec)^2} \\ & \times\left(-\frac{3}{4}-ec+\lambda e' c+\lambda e\omega' s-\frac{e^2}{4}+\frac{1}{2}\lambda ee'\right), \end{aligned} \tag{A13}$$

$$\lambda^{-2}s_{\phi\phi} = -\frac{es}{(1+ec)^2}. \tag{A14}$$